Homework 1

Math 126

Due September 17, 2021 by 5pm

Name:

Topics covered: complex numbers, arithmetic

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *(a) Show that for $n \geq 1$ and $z \neq 1$,*

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) Fix $a \in \mathbb{C}$. Prove[1] *that*

$$\left| \frac{z - a}{1 - \bar{a}z} \right| = 1$$

when $|z| = 1$.

Solution. □

Problem 2. *A quaternion is a 4-tuple of real numbers (a, b, c, d), typically written (in analogy with $\mathbb{C}$) as $a + bi + cj + dk$. Quaternions are added in the way you would guess; multiplication is defined using the relations $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$.*

(a) Prove that every nonzero quaternion has a multiplicative inverse.[2]

(b) Quaternion multiplication is not commutative. Determine all the quaternions that commute with i.

Solution. □

Problem 3. *Show that if n, m are integers that can be expressed as a sum of four square integers, then the product nm is also a sum of four squares. Is the same statement true if "four" is replaced by "three"?*

Solution. □

Problem 4. *Writing $z = x + iy$, sketch*[3] *the image of the following curves under the map $f(z) = z^2$.*

(a) $|z - 1| = 1$

(b) $x = 1$

(c) $y = x + 1$

(d) $y = 1/x$

Solution. □

Problem 5. *Let $p(z)$ be a polynomial, i.e. $p(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_i \in \mathbb{C}$. Assume $n \geq 1$.*

(a) Prove that if $p(a) = 0$ (we say a is a root of p), then there exists a polynomial $q(z)$ so that $p(z) = (z - a)q(z)$. [4]

[1] This computation can be painless. It will come up later in the course.

[2] Hint: follow as closely as possible the proof from class in the complex case.

[3] One motivation for this problem is for you to learn how to put pictures in your homework. Ask me if you have questions about how to do this.

[4] Hint: one way to do this uses the factoring $z^n - a^n = (z - a)(z^{n-1} + az^{n-2} + \cdots + a^{n-2}z + a^{n-1})$.

(b) Prove that given any a (not necessarily a root), there exists a polynomial q so that $p(z) = p(a) + (z-a)q(z)$.

(c) Find $q(z)$ when $p(z) = 1 + z + z^2 + \cdots + z^n$, and $a = -1$.

Solution. □

Problem 6. *Consider the complex exponential as a function* $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. *Is* $\exp$ *injective? surjective? Repeat for the real exponential function* $\exp : \mathbb{R} \to \mathbb{R} \setminus \{0\}$*?* [5]

Solution. □

[5] We will define the exponential map as a power series by Lecture 2 (at the latest). Usually I try to make the homework that's due a given week only based on lectures from the previous week(s), but that is difficult for the first assignment.