Homework 3

Math 242

Due March 20, 2020 by 5pm

Topics covered: Hurewicz theorem, cohomology, Hom functor, Ext groups, cobordism

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *Compute* $\pi_2(\mathbb{R}P^2 \vee \mathbb{R}P^2)$.

Solution. □

Problem 2. *Let* M *be an abelian group. Show that* $\operatorname{Hom}(-, M)$ *preserves split short exact sequences of abelian groups, i.e. if* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is a split short exact sequence, then*

$$0 \to \operatorname{Hom}(C, M) \xrightarrow{p^*} \operatorname{Hom}(B, M) \xrightarrow{i^*} \operatorname{Hom}(A, M) \to 0$$

is also a split short exact sequence.

Solution. □

Problem 3. *Compute* $\operatorname{Ext}^k(\mathbb{Z}, \mathbb{Z})$ *with the free resolution defined in class*

$$\cdots \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

Solution. □

Problem 4. *True or false: the functors* $h_k(X, A) = \operatorname{Hom}(H_k(X, A), \mathbb{Z})$ *do not define a (co)homology theory on the category of pairs of spaces.*[1]

Solution. □

Problem 5. *Let* (X, x_0) *be a connected cell complex. Show that every homomorphism* $\phi : \pi_1(X, x_0) \to \mathbb{Z}$ *is induced by a map* $f : (X, x_0) \to (S^1, s_0)$ *that is unique up to homotopy fixing* x_0. [2]

Solution. □

Problem 6 (Hatcher 3.1.13)**.** *Let* X *be a cell complex, and recall* $[X, S^1]$ *denotes the set of homotopy classes of based maps* $(X, x_0) \to (S^1, s_0)$. *Prove the map* $[X, S^1] \to H^1(X; \mathbb{Z}) \cong \operatorname{Hom}(H_1(X), \mathbb{Z})$ *defined by sending* $[f] \mapsto f_*$ *is a bijection (group isomorphism).*

Solution. □

Problem 7. [3]

(a) *Prove that every oriented surface bounds a 3-manifold.*

(b) *Prove that if a surface* M *is a fiber bundle* $M \to S^1$ *with fiber* S^1, *then* M *bounds a 3-manifold. Which surfaces does this apply to?*

(c) *Prove that if if* M, N *are surfaces that differ by a surgery, then* M, N *are cobordant.*[4]

[1] Hint: this problem may be harder than it seems. Be rigorous!

[2] This problem may be a bit painful to write down precisely. I'm most interested to see that you have the main ideas.

[3] This problem could be hard. Please ask for help.

[4] Surgery on surfaces is the process of removing $S^1 \times D^1$ and replacing it with $D^2 \times S^0$ (or the inverse of this). Ask me, or see page 5 of Ranicki's *Algebraic and geometric surgery.*

(d) *Prove that $\mathbb{R}P^2$ does not bound a 3-manifold. Hint: Proceed by contradiction, and construct a closed 3-manifold whose Euler characteristic is odd; now look up the statement of Poincaré duality.*

Solution. □

Problem 8. *Prove that the cobordism group of 2-dimensional manifolds is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.* [5]

Solution. □

[5] Recall from the classification of surfaces that any surface is either S^2, a connected sum of tori $T^2\#\cdots\#T^2$, or a connected sum $\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$. How does the Klein bottle fit into this classification?