Homework 2

Math 242

Due February 14, 2020 by 5pm

Topics covered: H-groups, homotopy groups, fibrations, LES of a fibration

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *Finish the proof of the H-group theorem. Show that the multiplication μ defined by $[\mu] = [p_1] \cdot [p_2] \in [Y \times Y, Y]$ is associative up to homotopy and has inverses up to homotopy.*

Solution. □

Problem 2. *Prove that there is no multiplication on $\mathbb{R}^3$ that makes it into a field.*[1][2]

Solution. □

Problem 3.

(a) *True or false: if $p : E \to B$ is a fibration, then p is surjective.*

(b) *Give an example of a surjective map $q : \mathbb{R}^2 \to \mathbb{R}^2$ that is not a fibration.*

Solution. □

Problem 4. *Let (B, b_0) be any based space. Let $PB = (B, b_0)^{(I,0)}$ denote the path space. Show that the map $p : PB \to B$ given by evaluation $p(f) = f(1)$ is a fibration. Do this by solving the lifting problem explicitly.*[3]

Solution. □

Problem 5. *Show that the evaluation map $p : \mathrm{SO}(n+1) \to S^n$ defined by $A \mapsto Ae_{n+1}$ has local sections.*[4]

Solution. □

Problem 6. *Compute all the homotopy groups of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty = \bigcup \mathbb{C}P^n$.*

Solution. □

Problem 7. *Recall that the special unitary group is defined as $\mathrm{SU}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^*A = I\}$, where A^* denotes conjugate transpose.*

(a) *Prove that $A \in \mathrm{SU}(2)$ can be expressed as*

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Use this to prove that $\mathrm{SU}(2)$ is homeomorphic to S^3.

(b) *Prove that there is a 2-fold cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$.*

(c) *Compute $\pi_2(\mathrm{SO}(n))$ for $n \geq 3$*

[1] Hint: construct a nowhere vanishing vector field on S^2.

[2] Further hint: try fixing $u \in \mathbb{R}^3$ and defining vector field $F(x) = ux$. This won't quite work – how can you fix it?

[3] There is a general (non-explicit) argument given in Bredon VII.6 using the homotopy extension property, but I'd like you to give a direct argument.

[4] Suggestion: do the case $n = 2$ in a way that will generalize to arbitrary n.

Solution. □

Problem 8 (Bonus). *In the spirit of the assignment's due date, write a topology poem.*

Solution. □