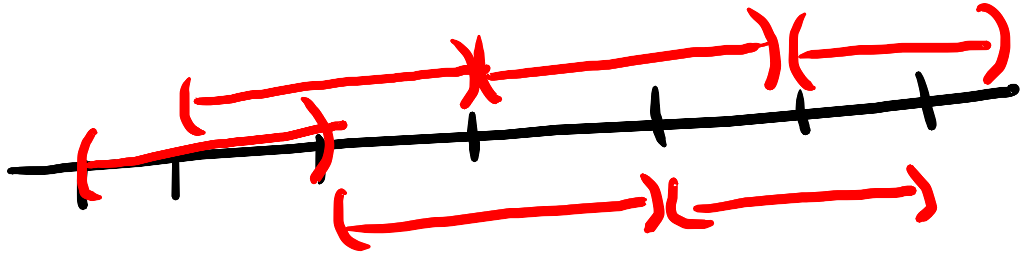
I. Properties of compact spaces

<u>Defn</u> A space X is <u>compact</u> if every open cover of X has a finite subcover.

<u>Ex</u> $\mathbb{R}$ is not compact. Since $U_n = (n, n+2)$ cover $\mathbb{R}$ but have no finite subcover

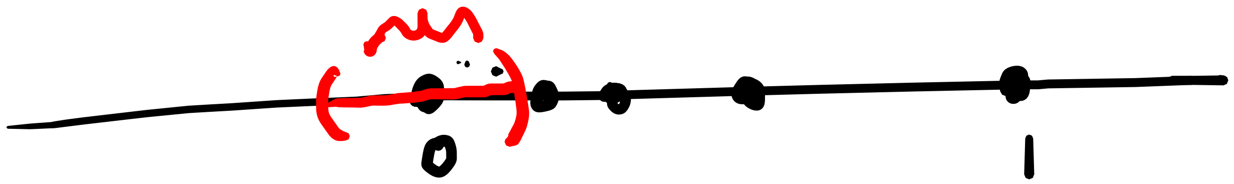


<u>Note</u> $\mathbb{R} = (-\infty, 1) \cup (-1, \infty)$ so $\mathbb{R}$ has a finite cover.

Ex $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$

is compact.

contains all but finitely many points of X.



Let $\mathcal{U}$ be any open of X.

$\exists\, U_0 \in \mathcal{U}$ s.t. $0 \in U_0$. U_0 open $\Rightarrow \exists\, r > 0$ s.t. $(-r, r) \subset U_0$.

U_0 contains all but finitely many pts of X. $\{x_1, \dots, x_m\}$.

Choose $U_i \in \mathcal{U}$ $x_i \in U_i$. Then $\mathcal{U}' = \{U_0, \dots, U_m\}$ finite subcover of $\mathcal{U}$.

Thm (Heine-Borel) $X \subset \mathbb{R}^n$ compact

$\iff$ X closed and bounded.

Examples

(1) $[0,1] \subset \mathbb{R}$ is compact

not, closed

(2) $\mathbb{Q}^2 \subset \mathbb{R}^2$ not compact (not bounded)

$\mathbb{Q}^2 \cap [0,1]^2$ not compact (not closed)

$\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ not compact (not bounded)

Prop (image of compact is compact)

$f: X \to Y$ continuous, X compact.

Then $f(X) = \{ y \in Y \mid y = f(x) \text{ some } x \in X \} \subset Y$ compact.

Proof. Take $\mathcal{U}$ open cover of $f(X)$.

Then $\mathcal{V} = \{ f^{-1}(U) : U \in \mathcal{U} \}$ open cover of X.

X compact $\Rightarrow$ $\exists$ finite subcover $\mathcal{V}' \subset \mathcal{V}$ of X

$$\mathcal{V}' = \{ f^{-1}(U_1), \ldots, f^{-1}(U_n) \}.$$

$\Rightarrow$ $\mathcal{U}' = \{U_1, \ldots, U_n\} \subset \mathcal{U}$ finite subcover of $f(X)$.

$f(x) \in f(X)$ $\quad x \in f^{-1}(U_i)$ $\Rightarrow$ $f(x) \in U_i$ $\quad \square$.

<u>Application</u> $S^1 \subset \mathbb{R}^2$ is compact

(assuming $[0,1]$ compact)

$$[0,1] \xrightarrow{f} S^1$$
$$t \longmapsto (\cos 2\pi t, \sin 2\pi t)$$

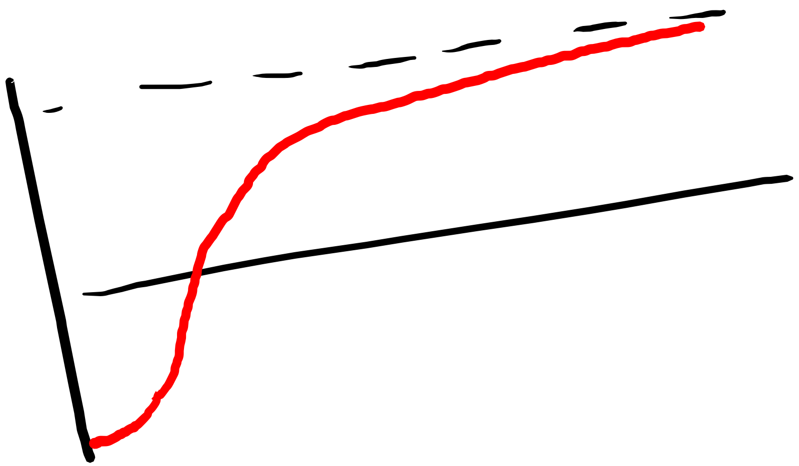
Prop (Boundedness & max value thms)

X compact, $f: X \to \mathbb{R}$ continuous. Then

(i) f is bounded, ie $\exists\ M$ s.t. $\|f(x)\| < M \quad \forall x \in X$.

(ii) f achieves a max value, ie $\exists\ a \in X$ s.t.

$$f(x) \le f(a) \quad \forall\ x \in X.$$



not compact.

$f: [0,\infty) \longrightarrow \mathbb{R}$ bound but doesn't achieve max value.

Prop (Boundedness & max value thms)

X compact, $f : X \to \mathbb{R}$ continuous. Then

(i) f is bounded, ie $\exists\ M$ s.t. $|f(x)| < M \quad \forall x \in X$.

Proof • a continuous function is locally bounded: for $p \in X$

$\exists$ open $U_p \ni p$ and M_p s.t. $|f(x)| < M_p \quad \forall x \in U_p$

(f continuous $\Rightarrow \exists$ open $U_p \ni p$ s.t. $x \in U_p \Rightarrow f(p)-1 < f(x) < f(p)+1$)

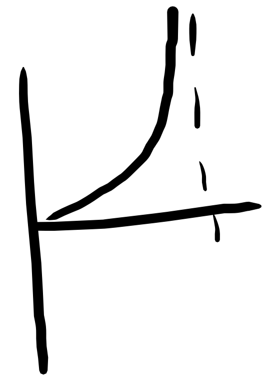
$M_p = \max\{|f(p)-1|, |f(p)+1|\}$

• $X = \bigcup_{p \in X} U_p$ open cover

X compact $\Rightarrow \exists\ p_1, \ldots, p_s$ s.t. $X = U_{p_1} \cup \cdots \cup U_{p_s}$.

Set $M = \max\{M_{p_1}, \dots, M_{p_s}\}$.

Then $|f(x)| \le M$. $\forall x \in X$. $\square$.



<u>Ex</u>. $\frac{x}{1-x}$ on $(0,1)$ unbounded.

not such function $\exists$ on $[0,1]$.

II. Heine-Borel

Thm $X \subset \mathbb{R}^n$ compact $\iff$ X closed & bounded.

Today: prove $(\Rightarrow)$

Fix $X \subset \mathbb{R}^n$ compact.

- X is bounded: note $X \subset \bigcup_{p \in X} B_1(p)$. X compact $\Rightarrow$

  $X \subset B_1(p_1) \cup \cdots \cup B_1(p_s)$

  some $p_1, \ldots, p_s \in X$.

  $\Rightarrow X \subset B_r(0)$ for some $r \gg 0$.

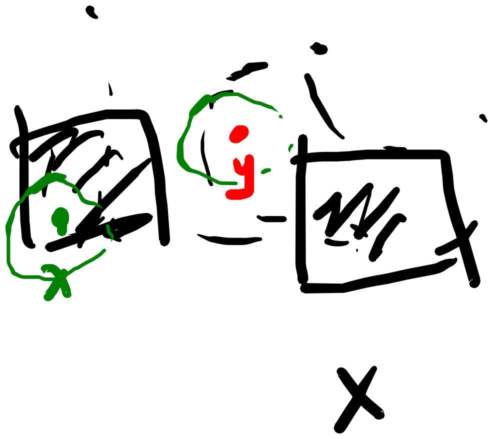
- X closed: show X^c open

Fix $y \in X^c$. (WTF: U open $y \in U \subset X^c$)

For $x \in X$ choose $r_x > 0$ s.t. $B_{r_x}(x) \cap B_{r_x}(y) = \emptyset$

$$X \subset \bigcup_{x \in X} B_{r_x}(x)$$

X compact $\Rightarrow$ $X \subset B_{r_{x_1}}(x_1) \cup \cdots \cup B_{r_{x_s}}(x_s)$



set $r = \min\{r_{x_1}, \ldots, r_{x_s}\}$

Then $B_r(y)$ disjoint from $B_{r_1}(x_1), \ldots, B_{r_{x_s}}(x_s)$

hence disjoint from X. $\square$

__Cor__ (boundedness Thm)

$f : X \to \mathbb{R}$ cts, X compact $\Rightarrow$ f bounded

__Pf__ X compact $\Rightarrow$ $f(X) \subset \mathbb{R}$ compact

$\Rightarrow$ $f(X)$ bounded ie $f(X) \subset [-M, M]$

$\Rightarrow$ $|f(x)| \leq M$ $\forall x \in X$. $\square$.

Next time: converse

X closed + bounded $\Rightarrow$ X compact.

$\cap$

$\mathbb{R}^n$