I. Product topology

<u>Problem</u> Given spaces X, Y.

Want topology on $X \times Y$.

<u>Naive guess</u> open set of $X \times Y$ are sets of form $U \times V$ where $U \subset X$, $V \subset Y$ open.

This doesn't work: eg $X = Y = \mathbb{R}$

problem: unions of open rectangles not necessarily open rectangles

Last time. A collection $\mathcal{B}$ of open sets of X is a basis if every open set is a union of elements of $\mathcal{B}$.

Lemma (defining a topology by a basis)

X set. $\mathcal{B} \subset P(X)$ collection of subsets of X.

If (i) $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2 \in \mathcal{B}$

(ii) $\bigcup_{B \in \mathcal{B}} B = X$

then $\mathcal{B}$ is a basis for a topology on X.

Lemma (defining a topology by a basis)

X set. $\mathcal{B} \subset P(X)$ collection of subsets of X.

If (i) $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2 \in \mathcal{B}$

(ii) $\bigcup_{B \in \mathcal{B}} B = X$

then $\mathcal{B}$ is a basis for a topology on X.

Pf: Define $U \subset X$ open if U is a union of elements of $\mathcal{B}$.

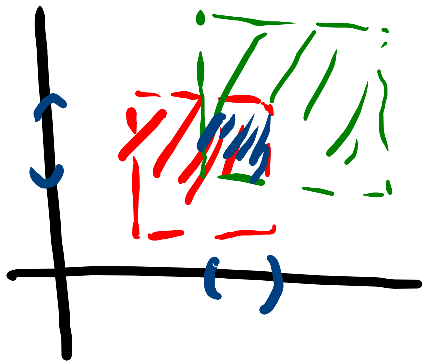
- X, ϕ open
- union ✓
- finite intersections

$$\left(\bigcup_i B_i\right) \cap \left(\bigcup_j B_j\right) = \bigcup_{i,j} (B_i \cap B_j) \quad ✓$$

$\square$

Application: X, Y spaces. Define

$$\mathcal{B} = \left\{ U \times V \;\middle|\; \begin{matrix} U \subset X \text{ open} \\ V \subset Y \text{ open} \end{matrix} \right\}.$$



observe $(U_1 \times V_1) \cap (U_2 \times V_2) =$

$$(U_1 \cap U_2) \times (V_1 \cap V_2)$$

$\Rightarrow \mathcal{B}$ satisfies assumptions of lemma. The corresponding top. is called the <u>product topology</u> on $X \times Y$

<u>Exercise</u>: projection maps $p : X \times Y \longrightarrow X$, $(x, y) \longmapsto x$ continuous

Example $X = Y = \mathbb{R}$

product topology on $\mathbb{R}^2$ has basis open rectangles.

= Standard topology on $\mathbb{R}^2$

(a set is a union of open balls $\Leftrightarrow$ union of open rectangles)

II. Closed and bounded sets

<u>Recall</u>: one goal of topology is to find topological invariants

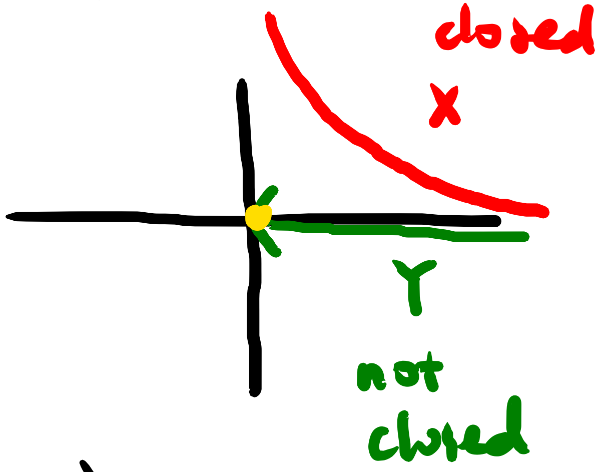
(i.e. a property "P" s.t. if $X \cong Y$ then X has "P" $\iff$ Y has "P")

<u>Ex</u>. Which of following is top. invariant for subsets of $\mathbb{R}^2$?

- closed not invariant
- finite invariant
- bounded ($C \subset \mathbb{R}^2$ <u>bounded</u> if $\exists\, r > 0$ st. $C \subset B_r(0)$)

not invariant

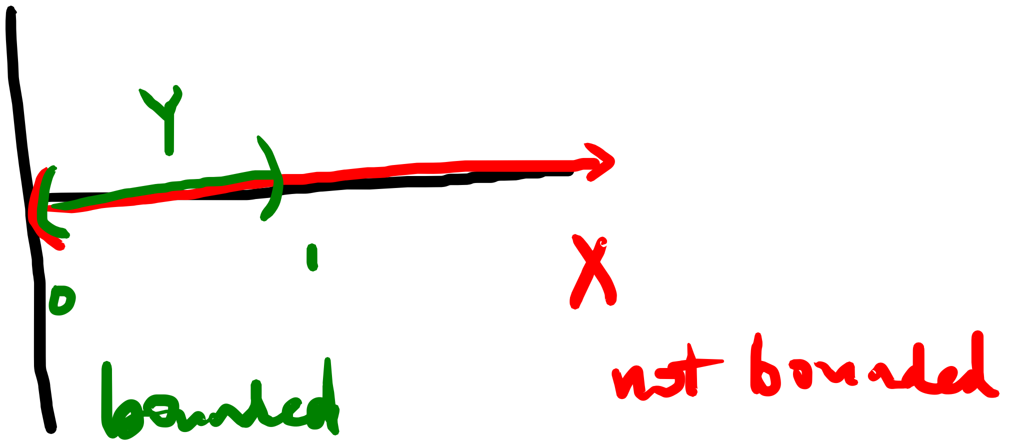
$X = \{(x, \frac{1}{x}) : x > 0\}$
$\cong$
$Y = \{(x, 0) : x > 0\}$



$X = \{(x,0) : x > 0\}$

$\cong$

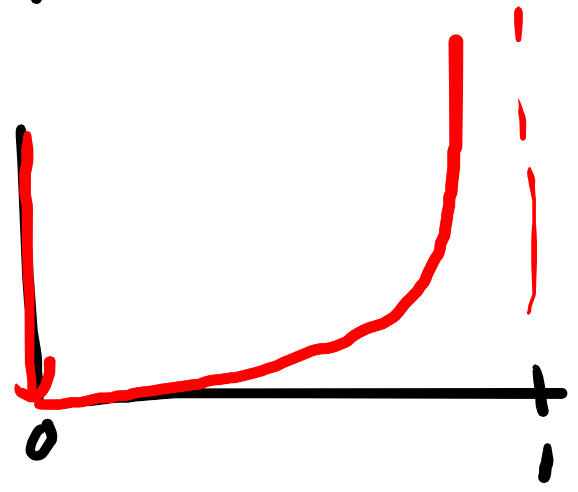
$Y = \{(x,0) : 0 < x < 1\}$



<u>Claim</u> $(0,1) \cong (0,\infty)$ top. equivalent.

eg $(0,1) \longrightarrow (0,\infty)$

$x \longmapsto \dfrac{x}{1-x}$



Thm "closed and bounded" is a topological invariant for subsets of $\mathbb{R}^n$.

To prove this we'll give a topological characterization of "closed & bounded"

III. Compactness

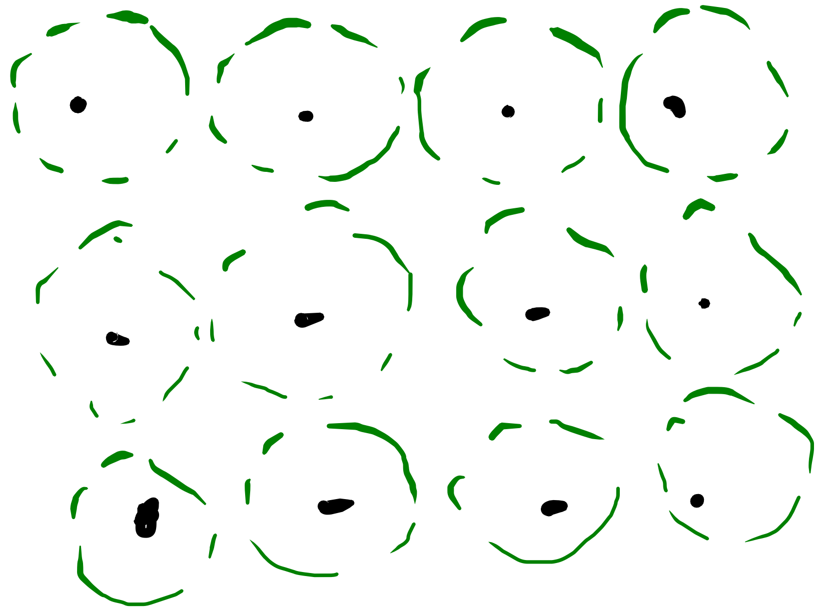
Fix $X \subset \mathbb{R}^n$

Defn A collection $\mathcal{U}$ of open sets in $\mathbb{R}^n$ is a <u>open cover</u> of X if $X \subset \bigcup_{U \in \mathcal{U}} U$. If $\mathcal{U}' \subset \mathcal{U}$ s.t. $X \subset \bigcup_{U \in \mathcal{U}'} U$ then say $\mathcal{U}'$ is a <u>subcover</u>.

A cover $\mathcal{U}$ is finite if it has finitely many elements.

Examples

- $X = \mathbb{Z}^2 \subset \mathbb{R}^2$



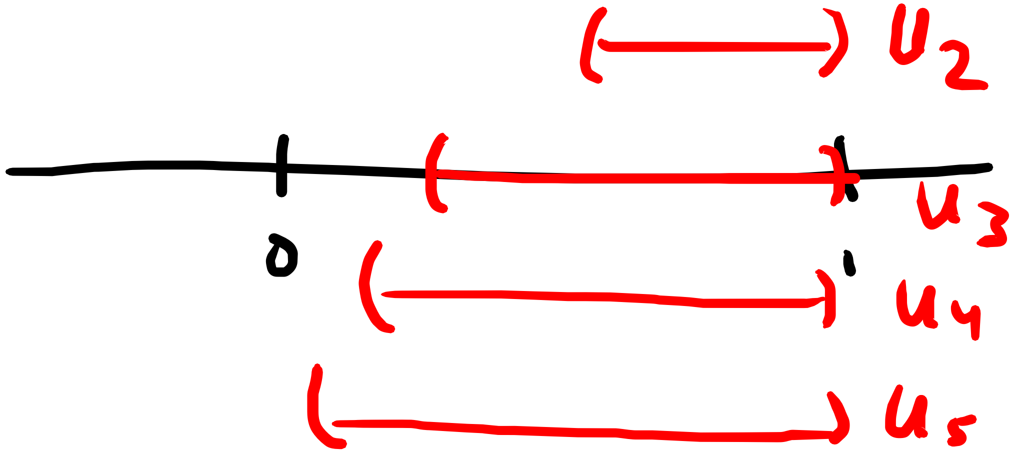
$\mathcal{U} = \{ B_{\frac{1}{2}}(v) : v \in \mathbb{Z}^2 \}$

open cover

No proper subset $\mathcal{U}' \subsetneq \mathcal{U}$ is an open cover so $\mathcal{U}$ has no subcovers.

- $X = (0,1) \subset \mathbb{R}$

$U_n = (\frac{1}{n}, 1)$ open cover of X

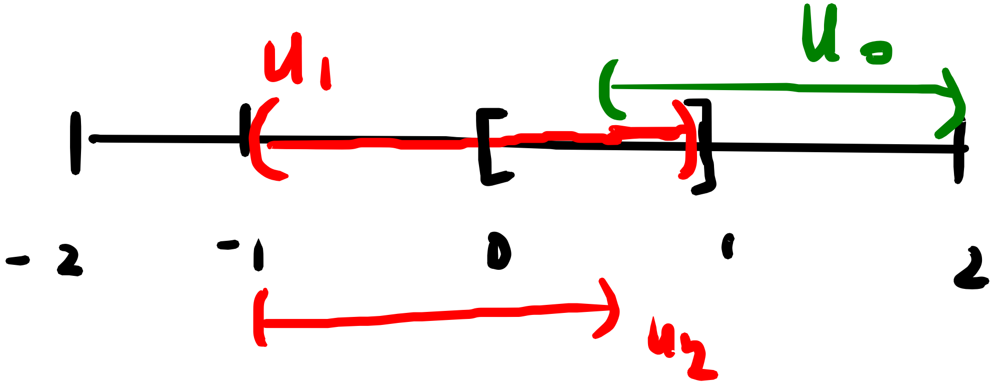


$\mathcal{U}' = \{ U_n : n > 100 \}$ is a subcover

$\mathcal{U}$ does not have a <u>finite</u> subcover

- $X = [0,1]$

$U_n = (-1, \frac{1}{n})$ $\quad U_0 = (\frac{1}{2}, 2)$ open cover



$\mathcal{U}$ has finite subcover

$\mathcal{U}' = \{U_0, U_1\}$.

Thm (Heine-Borel) $X \subset \mathbb{R}^n$. TFAE

(i) X is closed & bounded

(ii) every open cover of X has a finite subcover (X is compact)

topological

<u>Cor</u> $(0,1)$ & $[0,1]$ not top. equivalent.

not closed but bounded

closed & bounded.