I. Closed set & limit points

X topological space

Fix $A \subset X$ any subset, $x \in X$.

<u>Subset trichotomy</u>: Either

(i) $\exists$ open U s.t. $x \in U \subset A$ — x is an <u>interior point</u> of A

(ii) $\exists$ open U s.t. $x \in U \subset A^c = X \setminus A$ — x is an <u>exterior pt</u>

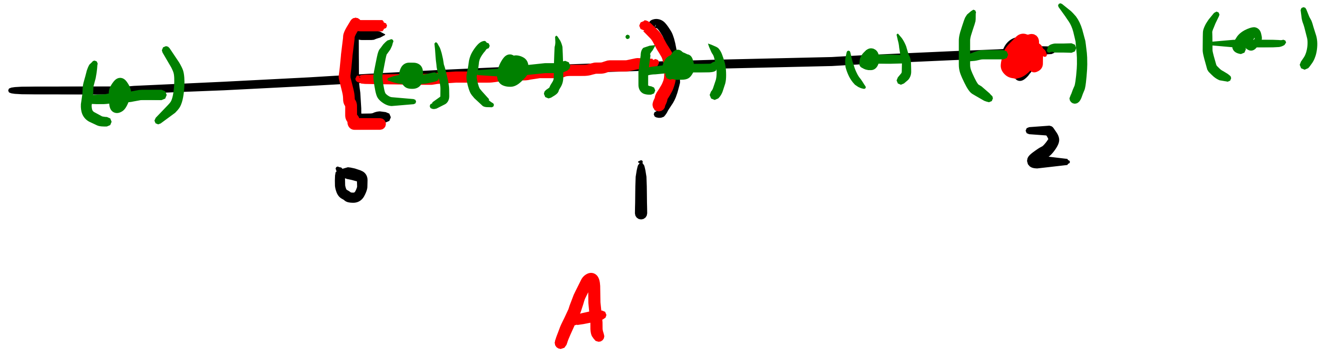
(iii) every open U containing x intersects both A & A^c.

then either (a) $\exists$ open U s.t. $U \cap A = \{x\}$ — x is an <u>isolated point</u>

or (b) every open U containing x contain a point of $A \setminus \{x\}$. — x is a <u>limit point of A</u>.

Examples

- $X = \mathbb{R}$ $\quad A = [0,1) \cup \{2\}$



interior points $(0,1)$

exterior points $(-\infty, 0) \cup (1,2) \cup (2,\infty)$

isolated points $\{2\}$

limit points $\{0,1\}$.

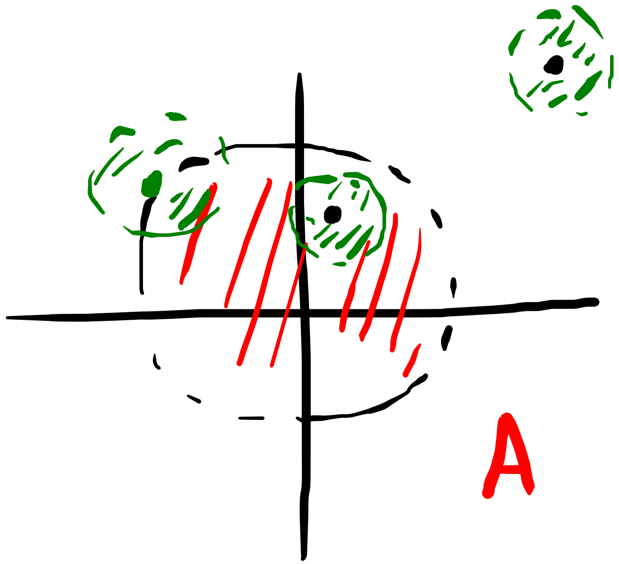
<u>Note</u> limits points can be in A or A^c.

- $A = B_1(0) \subset \mathbb{R}^n$

interior points A

exterior points $\{x \in \mathbb{R}^n : |x| > 1\}$

limit points $\{x \in \mathbb{R}^n : |x| = 1\}$.



Lemma X metric space, $A \subset X$

x limit point of A $\iff$ $\exists$ sequence $(a_n) = a_1, a_2, \ldots \in A \setminus \{x\}$
that **converges** to x

↳ $\forall \varepsilon > 0$ $\exists$ N st. for $n > N$
$d(a_n, x) < \varepsilon$

Proof

$(\Rightarrow)$ Assume x limit pt of A. Then $B_{\frac{1}{n}}(x)$ contains some pt $a_n \in A \setminus \{x\}$.

Then (a_n) converges to x.

$(\Leftarrow)$ Assume $\exists$ $(a_n) \subset A \setminus \{x\}$ converging to x.
Fix U open containing x. $\exists\, \varepsilon > 0$ st. $B_\varepsilon(x) \subset U$. $\exists\, N > 0$ st.
$n > N \Rightarrow a_n \in B_\varepsilon(x) \subset U$ $\square$

Ex $X = \mathbb{R}$, $A = \mathbb{Q}$

every $x \in \mathbb{R}$ is a limit point of A.

write $x = n.x_1 x_2 x_3 \ldots$ decimal form

$a_1 = n.x_1$, $a_2 = n.x_1 x_2$, $\ldots$ sequence in $\mathbb{Q}$ converging to x.

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observation: $A \subset X$ open $\Longleftrightarrow$ every point of A is an interior pt.

$(\Rightarrow)$ immediate: $x \in A$ take $U = A$

$(\Leftarrow)$ for $x \in A$ $\exists$ U_x open $x \in U_x \subset A$ Then

$$A = \bigcup_{x \in A} U_x$$ union of open sets, hence open.

Defn. Say $A \subset X$ is **closed** if it contains all of its limit points

Ex. $[0,1) \cup \{2\} \subset \mathbb{R}$ not closed b/c doesn't contain 1 (limit pt)

OTOH $[0,1] \cup \{2\}$ is closed.

Rmk Sets are not doors.

eg $[0,1) \subset \mathbb{R}$ neither open or closed.

X, ϕ both open and closed.

Prop For $A \subset X$, A closed $\Leftrightarrow$ A^c open.

Proof ($\Rightarrow$) A closed. Fix $x \in A^c$.

WTS: $\exists$ open U w/ $x \in U \subset A^c$.

by subset trichotomy $x \in A^c$ either exterior pt or limit pt.

A closed $\Rightarrow$ x not limit pt $\Rightarrow$ x exterior ✓

($\Leftarrow$) A^c open. $\Rightarrow$ every $x \in A^c$ is an exterior pt of A.

$\Rightarrow$ A contains all its limit pts.

Exercise: $f: X \to Y$ continuous $\Longleftrightarrow$ $f^{-1}(A)$ closed for $A \subset Y$ closed.

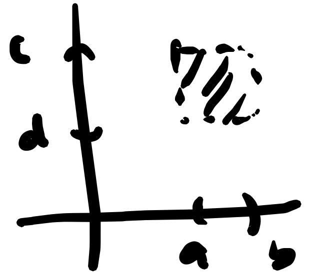
II. Basis for a topology

Defn X top. space. $\mathcal{U} \subset P(X)$ open sets.

$\mathcal{B} \subset \mathcal{U}$ is a basis if every open set is a union of open sets in $\mathcal{B}$. if $U \in \mathcal{U}$ $\exists$ $\{B_\alpha\} \subset \mathcal{B}$ s.t.

$$U = \bigcup B_\alpha .$$

Ex. $X = \mathbb{R}^2$. open balls $B_r(x)$ forms a basis $r > 0$ $x \in \mathbb{R}^2$.



- open rectangles $(a,b) \times (c,d)$ also form a basis
- $\mathcal{B} = \{ B_r(x) : x \in \mathbb{Q}^2, r \in \mathbb{Q} \}$ also a basis (countable)

Prop X, Y spaces. $\mathcal{B}$ basis for Y.

$f : X \to Y$ continuous $\iff$ $f^{-1}(B)$ open for $B \in \mathcal{B} \subset \mathcal{U}$

Proof: $(\Rightarrow)$ immediate

$(\Leftarrow)$ Let $U \subset Y$ open. Write $U = \bigcup B_\alpha$ $\quad B_\alpha \in \mathcal{B}$

Then $f^{-1}(U) = f^{-1}(\bigcup B_\alpha) = \bigcup f^{-1}(B_\alpha)$ union of open sets hence open $\square$

Prop X, Y spaces. $\mathcal{B}$ basis for Y.

$f: X \to Y$ continuous $\iff$ $f^{-1}(B)$ open for $B \in \mathcal{B} \subset \mathcal{U}$

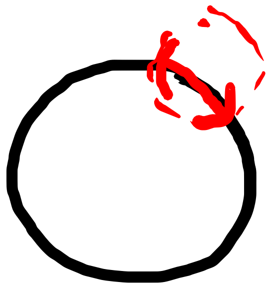
Application exponential map $f: \mathbb{R} \longrightarrow S^1 \subset \mathbb{R}^2 = \mathbb{C}$

$t \longmapsto (\cos t, \sin t) = e^{it}$

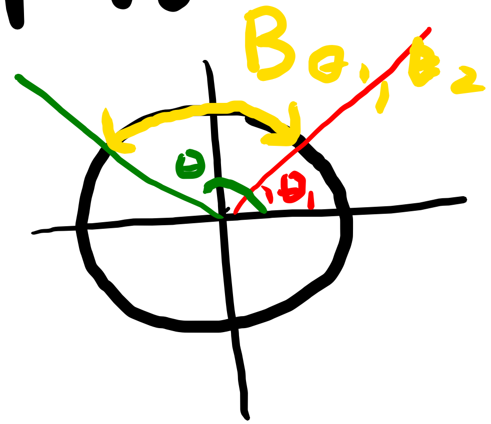
Claim f continuous.

Pf: Basis for topology on S^1: for $\theta_1 < \theta_2$ in $\mathbb{R}$

$\theta_2 - \theta_1 < 2\pi$



$B_{\theta_1,\theta_2} =$



show $f^{-1}(B_{\theta_1,\theta_2})$ open.

$$f^{-1}(B_{\theta_1,\theta_2}) = \bigcup_{k\in\mathbb{Z}} (\theta_1 + 2\pi k, \theta_2 + 2\pi k)$$

union of open sets hence open