I. Metric spaces

<u>Recall</u> <u>topological space</u> is a set X

and subset $\mathcal{U} \subset P(X) = \{\text{subsets of } X\}$ s.t.

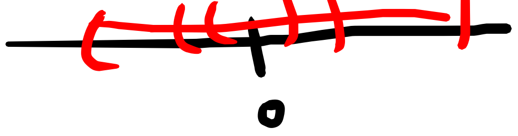
"open sets"

- $X, \phi \in \mathcal{U}$
- if $U_\alpha \in \mathcal{U}$ for α in some index set A

  then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{U}$.
- if $U_1, \dots, U_n \in \mathcal{U}$ then $U_1 \cap \dots \cap U_n \in \mathcal{U}$.

<u>Rmk</u> intersection of infinitely many open sets need not be open

eg $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$ $\quad \bigcap U_n = \{0\}$ not open

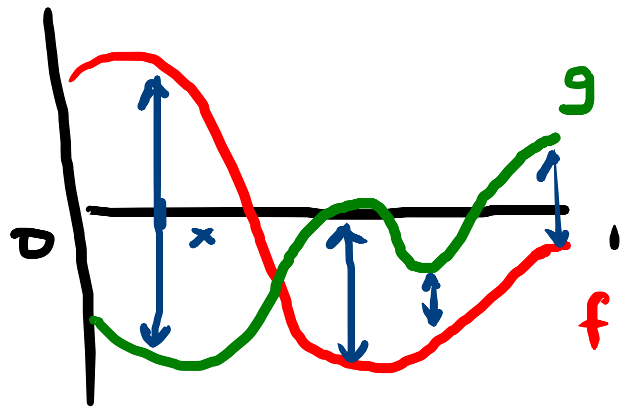


Ex (metric spaces)

Consider $X = \{$ continuous maps $f: [0,1] \to \mathbb{R} \}$

e.g. $f(x) = x^2$ $\quad g(x) = \sin(x)$ $\quad h(x) = |x - \tfrac{1}{2}|$ $\qquad f, g, h \in X$

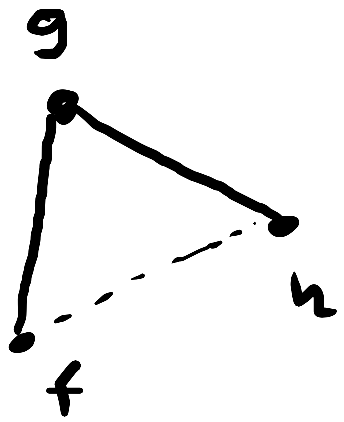
Define "distance" between $f, g \in X$



$$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$$

d satisfies same properties as the usual distance function on $\mathbb{R}^n$

- $d(f,g) \geq 0$ and $d(f,g) = 0 \iff f = g$
- $d(f,g) = d(g,f)$
- $d(f,h) \leq d(f,g) + d(g,h)$



- Say $U \subset X$ is open if for each $f \in U$
$\exists\ r > 0$ s.t. $B_r(f) \subset U$

$$\begin{array}{c} B_r(f) \\ \parallel \\ \{ g \in X \mid d(f, g) < r \} \end{array}$$

This makes X into a topological space.

$U \subset X$

- Terminology: d is called a <u>metric</u>. (X, d) is a <u>metric space</u>

Any metric space is a topological space as above.
(Note: converse not true)

- Most important example (for us): $\mathbb{R}^n$ with metric
$$d(x, y) \equiv |x - y| = \left(\Sigma (x_i - y_i)^2 \right)^{1/2}$$

II. Continuity

<u>Recall</u> For X, Y spaces. $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U)$ is open in X for any open $U \subset Y$.

<u>Examples</u>

- polynomials $p(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$ $\quad a_i \in \mathbb{R}$

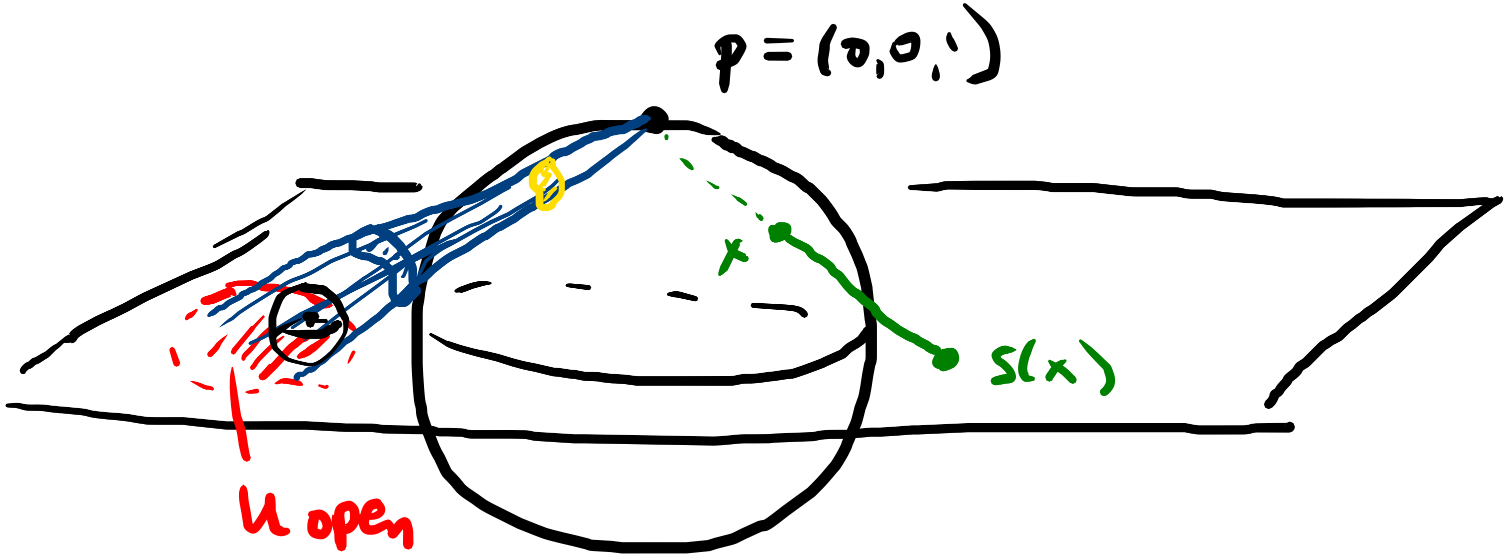
  defines $p: \mathbb{R} \longrightarrow \mathbb{R}$. p is continuous (calculus)

  similarly polys in several variable are continuous

$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto x^2 + xy - y + 7$$

- Stereographic projection

$s: S^2 \setminus p \longrightarrow \mathbb{R}^2$



Recall

subspace topology:

$V \subset \mathbb{R}^2 \subset \mathbb{R}^3$ open if

$V = U \cap \mathbb{R}^2 \quad U \subset \mathbb{R}^3$ open

Claim s is continuous

Pf: Fix $U \subset \mathbb{R}^2$ open. WTS $s^{-1}(U)$ open in $S^2 \setminus p$.

ie find open set $R \subset \mathbb{R}^3$ st $s^{-1}(U) = R \cap (S^2 \setminus p)$

$R =$ ray spanned by p & U. $= \{p + t(x-p) \mid x \in U, t > 0\} \subset \mathbb{R}^3$ open.

<u>Exercise</u> A composition of continuous functions is continuous.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad f, g \text{ continuous} \Rightarrow g\circ f \text{ continuous.}$$

<u>Pf</u>: Take $U \subset Z$ open

$$(g\circ f)^{-1}(U) = \{x \in X \mid (g\circ f)(x) \in U\}$$
$$g(f(x)) \in U$$

<u>Claim</u> $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

$$f^{-1}(g^{-1}(U)) = \{x \in X \mid f(x) \in g^{-1}(U)\}.$$
$$= \{x \in X \mid g(f(x)) \in U\}.$$

U open

g cts $\Rightarrow g^{-1}(U)$ open

f cts $\Rightarrow f^{-1}(g^{-1}(U))$ open

$$f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$$

✓

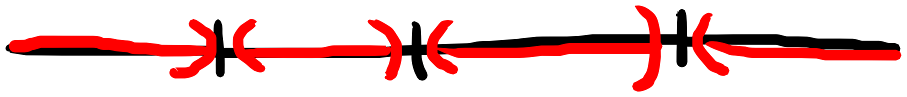
III. Topologies on $\mathbb{R}$

A given set has many different topologies

<u>Ex</u> $X = \mathbb{R}$

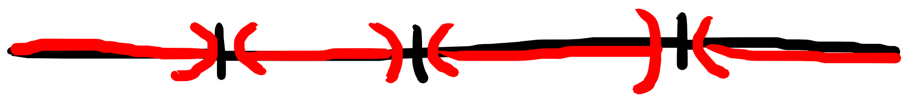
- standard topology : from metric $d(x,y) = |x-y|$
- discrete topology : every subset is open. in particular $\{x\}$ is open
- indiscrete topology : only open subsets are $X, \emptyset$.
- cofinite topology : $U \subset \mathbb{R}$ open if $\mathbb{R} \setminus U$ is finite

U open



III. Topologies on $\mathbb{R}$

- cofinite topology: $U \subset \mathbb{R}$ open if $\mathbb{R} \setminus U$ is finite



U open

<u>Rmks</u>
- this is an example of the Zariski topology in algebraic geometry.
- This topology is <u>not</u> induced from a metric!

(A) If (X, d) metric space then for any $p \neq q$ in X, $\exists$ open $p \in U$, $q \in V$ s.t. $U \cap V = \phi$. (Metric spaces are Hausdorff)

(B) in $\mathbb{R}$ w/ cofinite topology for any U, V open $U \cap V \neq \phi$.

- Proof of (A):

$V = B_{r/4}(q)$

$r = d(p,q)$

$U = B_{r/4}(p)$

- Proof of (B): Fix U, V open

$U \cap V = \phi \Rightarrow (U \cap V)^c = \mathbb{R}$

But $(U \cap V)^c = U^c \cup V^c$ is finite. $\Rightarrow \mathbb{R}$ is finite

(contradiction)