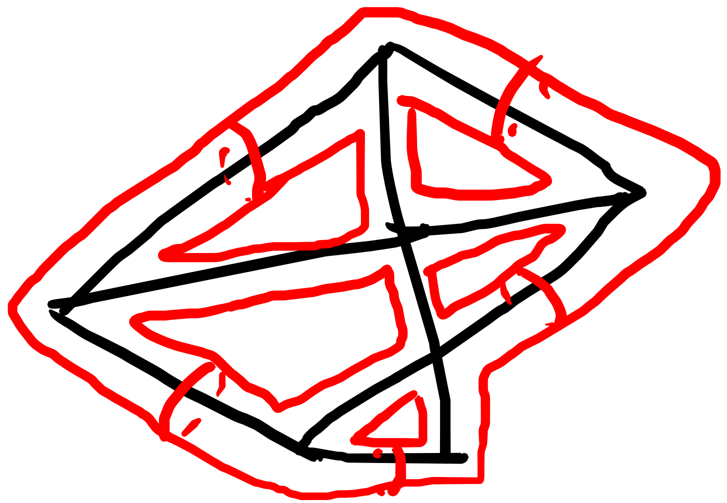
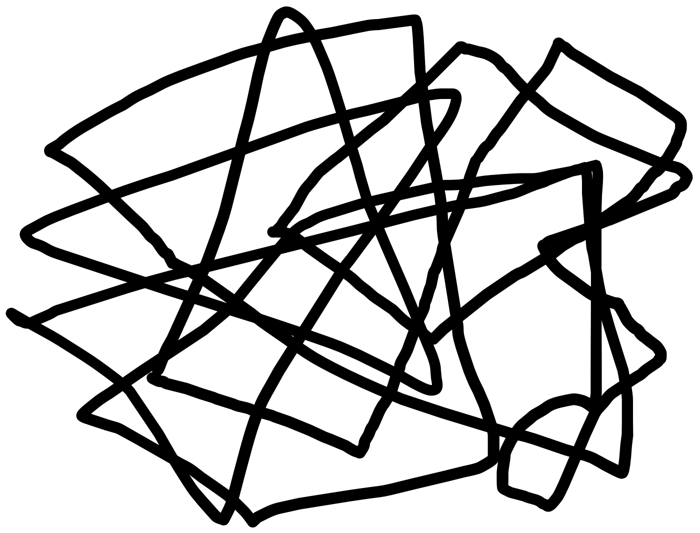
Finishing the classification of surfaces

<u>Thm</u> [Every closed surface is obtain from S^2 by annulus & Mobius attachments.]

<u>Ex</u>. Start w/ any finite graph $G \subset \mathbb{R}^3$. Thicken G to get $X \subset \mathbb{R}$. ∂X is a closed surface



G



Thm Every closed surface is obtain from S^2 by annulus & Mobius attachments.

Proof Let S be a closed surface.

(1) Triangulate $S \cong |K|$

(2) inductive argument: use surgery to replace K with another combinatorial surface $\hat{K}$ with $\chi(\hat{K}) > \chi(K)$:

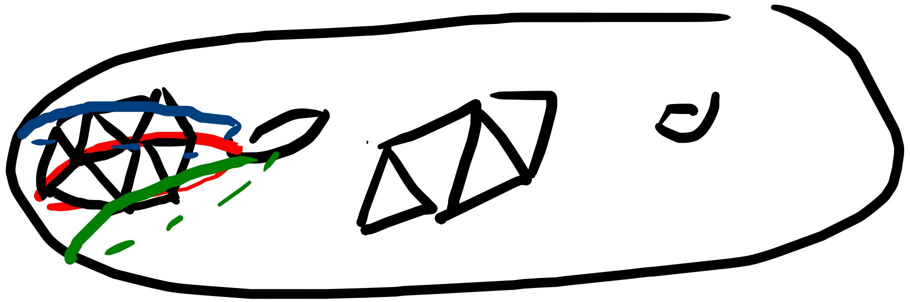
if $|K| \cong S^2$ we're done. Otherwise $\exists$ edge loop $\gamma \subset K^1$ that doesn't separate $|K|$

(Recall we proved

if every edge loop in K' separates, then

$|K| \cong S^2$.)

Then perform combinatorial surgery on γ

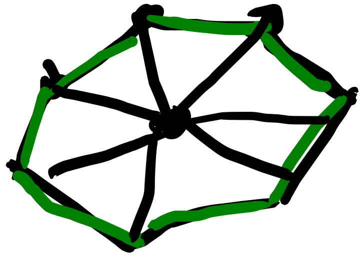


γ

remove

$N(\gamma) \cong A$ or M

and cone each ∂ comp by a disk. to get combinatorial surface $\hat{K}$



We proved $\chi(\hat{K}) > \chi(K)$.

We can repeat this inductively

This procedure stops when $\chi(\hat{K}) = 2$.

$\implies |\hat{K}| \cong S^2$

(3) Reverse surgery to conclude. $\square$

<u>Thm</u> Denote $M_g = \underbrace{T^2 \# \cdots \# T^2}_{g}$

$= S^2$ w/ g annulus attachments

$N_k = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k} = S^2$ w/ k Mobius attachments.

No two of S^2, M_g $(g \geq 1)$, N_k $(k \geq 1)$ are top. equivalent.

<u>Recall</u> $T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

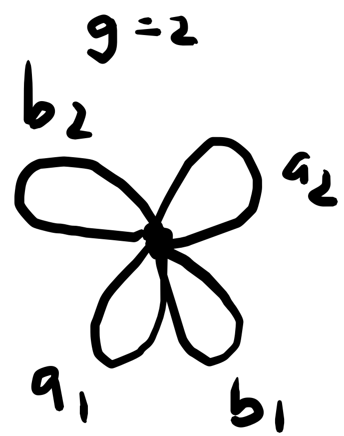
Proof of Thm (sketch)

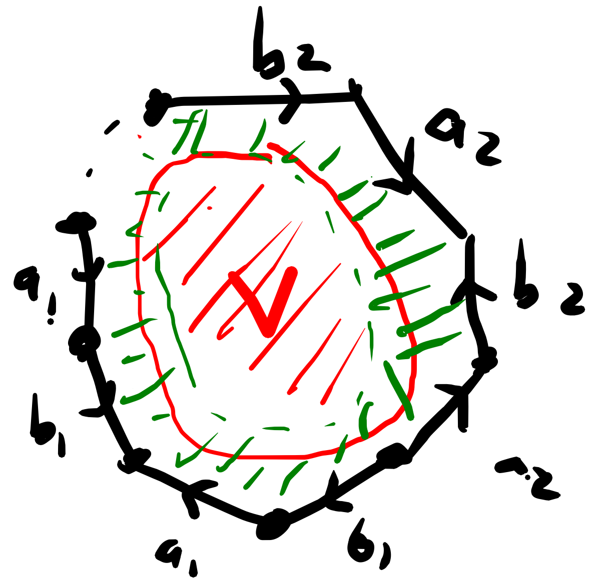
Compute π_1 and its abelianization

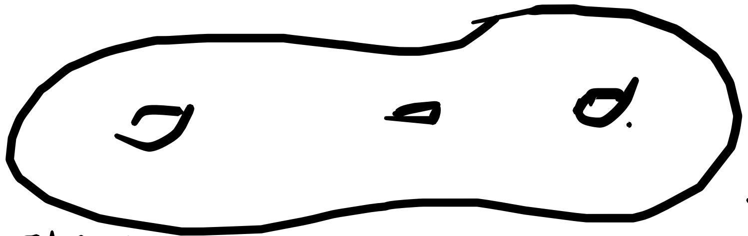
- $M_g \cong$ 4g-gon w/ side identifications

$$= U \cup V$$

$$U \simeq \bigvee_{2g} S^1, \qquad V \simeq D^2$$





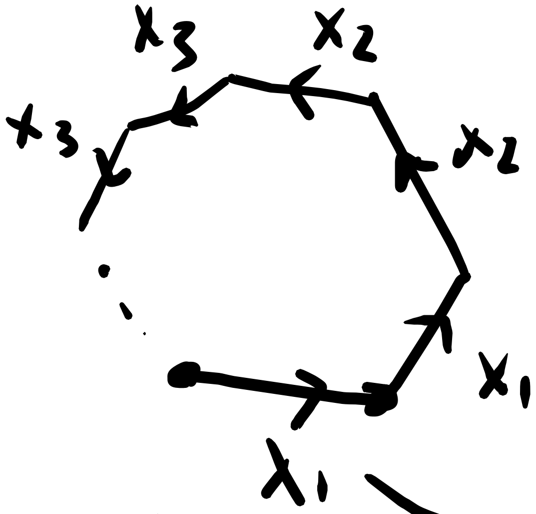


use van Kampen

$$\pi_1(M_g) = \langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$$

$$[a_1, b_1] \cdots [a_g, b_g] = 1$$

- $N_k = 2k \text{ gon} / \sim$ =



$\pi_1(N_k) = \langle x_1, \ldots, x_k \mid x_1^2 x_2^2 \cdots x_k^2 = 1 \rangle$

$\pi_1(S^2) = 1$

$\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$

$\pi_1(N_k) = \langle x_1, \ldots, x_k \mid x_1^2 x_2^2 \cdots x_k^2 = 1 \rangle$

Look at abelianization

$\pi_1(M_g)^{ab} \cong \mathbb{Z}^{2g}$

$\pi_1(N_k)^{ab} \cong \mathbb{Z}^{k-1} \times \mathbb{Z}/2\mathbb{Z}$

$\pi_1(S^2)^{ab} = \{0\}$

No two of these groups are isomorphic.

$\Rightarrow$ no two of S^2, M_g, N_k are top. equiv. $\square$

II. Surfaces w/ boundary

<u>Defn</u> Say a surface S is <u>nonorientable</u> if it contains a Mobius band. (eg N_k)

Otherwise say S is orientable.

if S is orientable then for any loop $\gamma \subset S$, a neighborhood $N(\gamma) \cong$ annulus.

Non-or surfaces are also called 1-sided.

Ex  is 2-sided

Rmk (Euler numbers)

- $\chi(S^2) = 2$
- $\chi(M_g) = 2-2g$

Pf 1 $M_g = T^2 \# \cdots \# T^2$

inductively

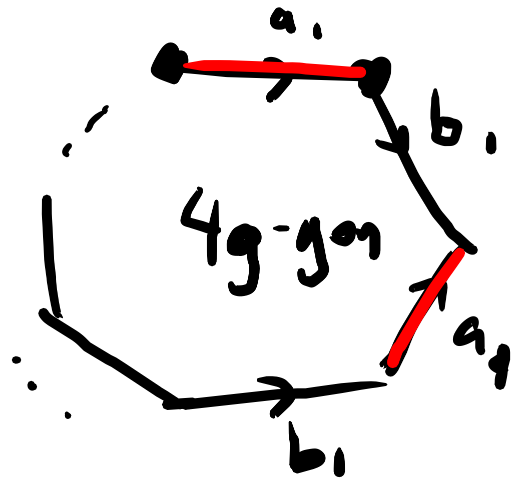
$\chi(K_1 \# K_2) = \chi(K_1) + \chi(K_2) - 2$

$\chi(T^2) = 0$

Pf 2 $M_g = (S^2 \setminus 2g \text{ disks}) \cup (g \text{ annulus})$

$$\chi(K_1 \cup K_2) = \underset{2-2g}{\chi(K_1)} + \underset{0}{\chi(K_2)} - \underset{0}{\chi(K_1 \cap K_2)}$$

Pf 3



$$\chi = V - E + F$$
$$= 1 - 2g + 1 = 2 - 2g$$

Similarly $\chi(N_k) = 2 - k$.

Ex $\chi(M_1) = \chi(N_2) = 0$

Rmk A closed surface is det by χ & orientability

S^2, M_g N_k

Similarly a surface w/ boundary is det. by χ, orientability & # ∂ components.

__Ex.__

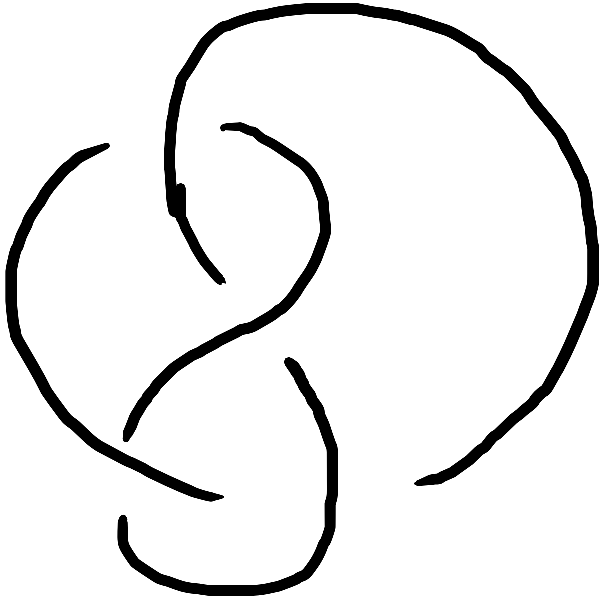
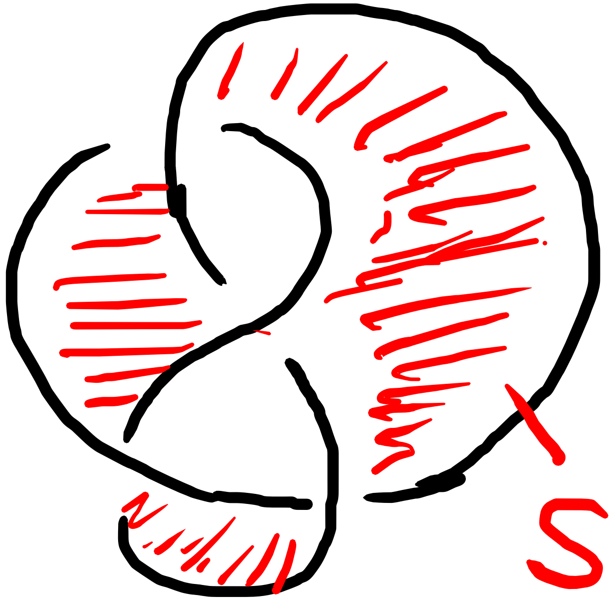


figure 8 Knot



__Q:__ What surface is this??

- $\partial S \cong S^1$ (1 boundary comp.)
- S is nonorientable

Similarly a surface w/ boundary is det. by χ, orientability & # ∂ components.

Ex.

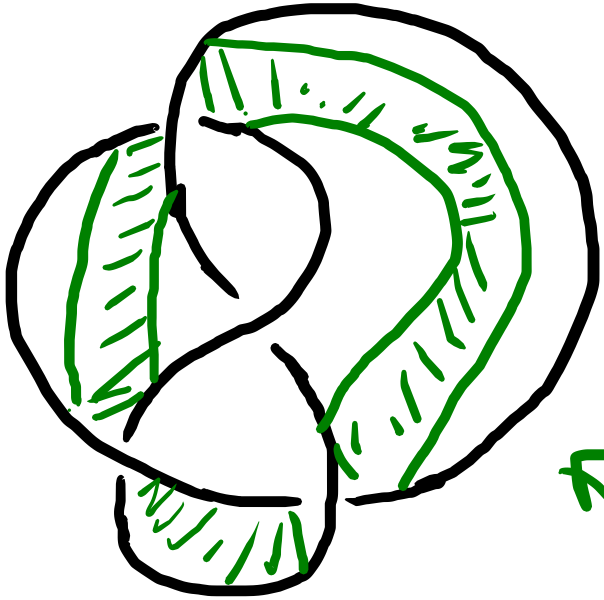
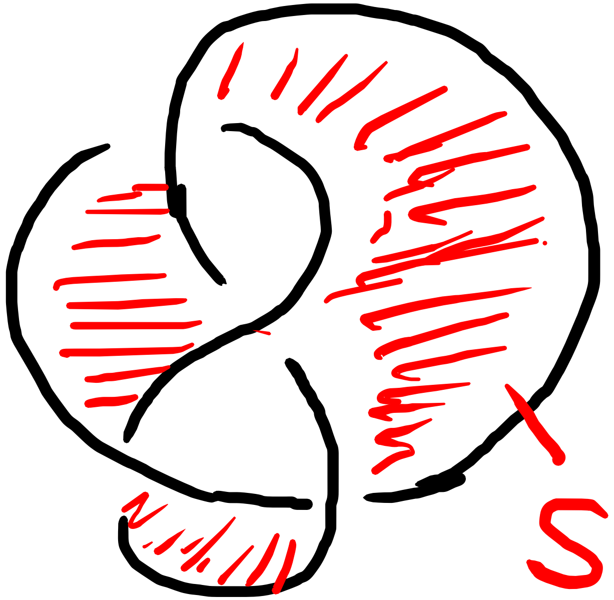


figure 8 Knot

Mobius



$S \subset \mathbb{R}^3$

Q: What surface is this??

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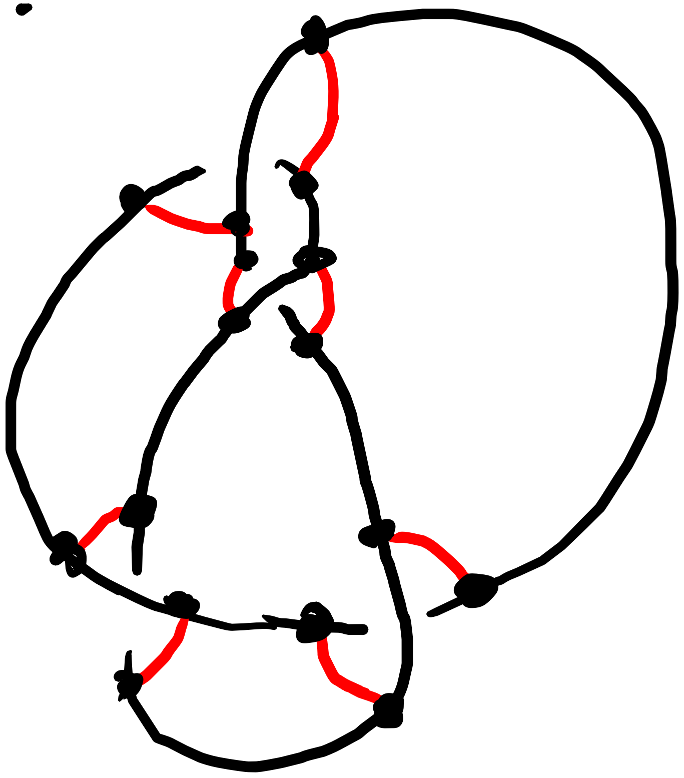
$\chi(S)$:



$\chi = V - E + F$

$= 16 - 24 + 7 = -1.$

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1 ∂ comp : consider $\hat{S} = S \cup_\partial \mathbb{D}^2$ closed surf.

$\chi(\hat{S}) = \chi(S) + 1 = 0. \Rightarrow \hat{S} =$ Klein bottle

$\Rightarrow S = K \setminus \mathbb{D}^2.$