I. Combinatorial characterization of S^2

<u>Thm</u> $|K|$ combinatorial surface

(a) Every edge loop in K' separates $|K|$ into two components

(b) $\chi(K) = 2$

(c) $|K| \cong S^2$

(a) $\Rightarrow$ (b) $\Rightarrow$ (c)

Consequently if $|K| \not\cong S^2$ then $\exists$ edge loop γ s.t. $|K| \setminus \gamma$ is connected. (surgery).

Thm $|K|$ combinatorial surface

(a) Every edge loop in K' separates $|K|$ into two components

(b) $\chi(K) = 2$

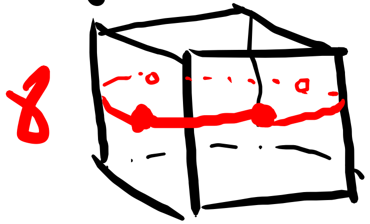
(c) $|K| \simeq S^2$

$(a) \Rightarrow (b) \Rightarrow (c)$

Proof Assume (a). $T \subset K$ max tree, G dual graph.

Claim G is a tree. (Then $\chi(K) = \chi(T) + \chi(G) = 2$.)

Suppose not. Then $\exists$ nonbacktracking loop γ in G.



By assumption γ separates $|K|$. One contains T. $\exists$ vertex in the other component ~~$\times$~~. ✓

Thm $|K|$ combinatorial surface

(a) Every edge loop in K' separates $|K|$ into two components

(b) $\chi(K) = 2$

(c) $|K| \cong S^2$

(a) $\Rightarrow$ (b) $\Rightarrow$ (c)

Proof Assume (b). T, G as before.

(b) $\Rightarrow$ G a tree. $|K|$ = union of neighborhoods of T & G

These neighborhoods are $\cong D^2$ and intersect in S^1.

$$\Rightarrow |K| \cong D^2 \cup_{S^1} D^2 \cong S^2$$

$\square$

Rmk. We proved (a) $\iff$ (b) $\implies$ (c).

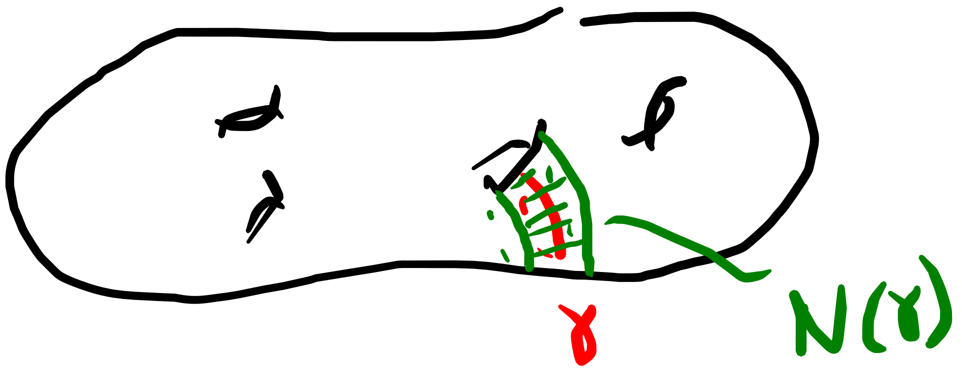
(c) $\implies$ (a) also true (Jordan curve theorem)

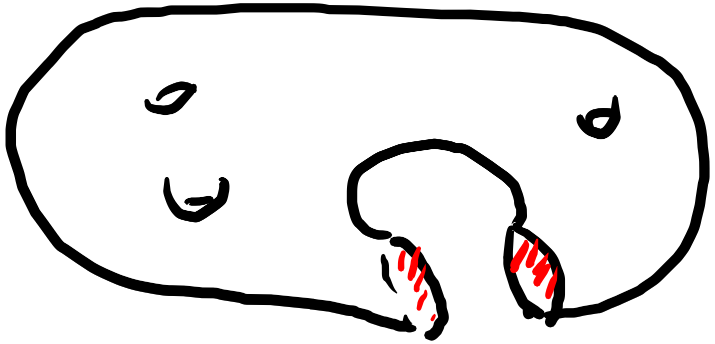
II. Surgery / Tubular neighborhoods

Surgery: input Surface S and a loop γ s.t. $S \setminus \gamma$ is connected.

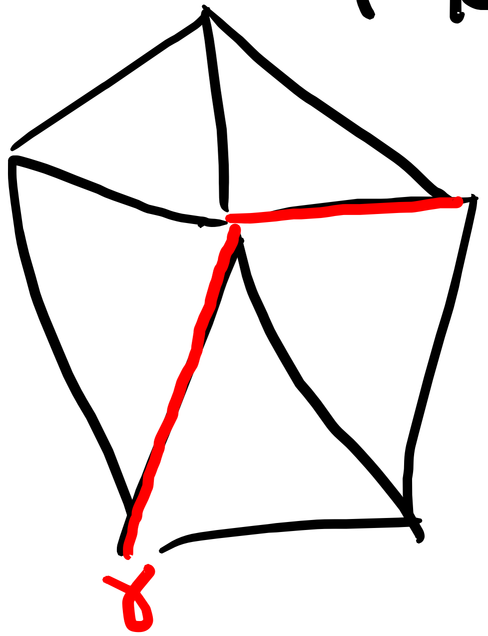
output: new surface S' w/ $\chi(S) < \chi(S')$.

$$S' = \left(S \setminus \operatorname{int} N(\gamma)\right) \cup \text{disks for each } \partial \text{ comp}$$



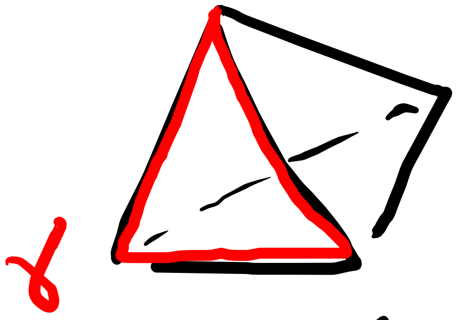


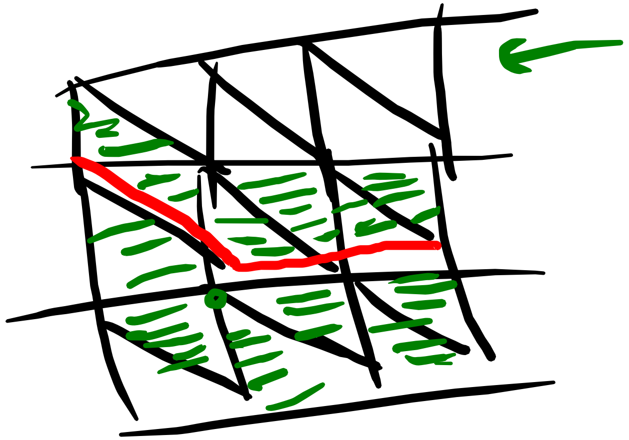
Need to define $N(\gamma)$ when $\gamma \subset K$ edge path (or even a subcomplex of 1-skeleton of K)



__Naive guess:__ take $N(\gamma)$ union of simplices that meet γ.

__Problem:__ $N(\gamma)$ may be too large

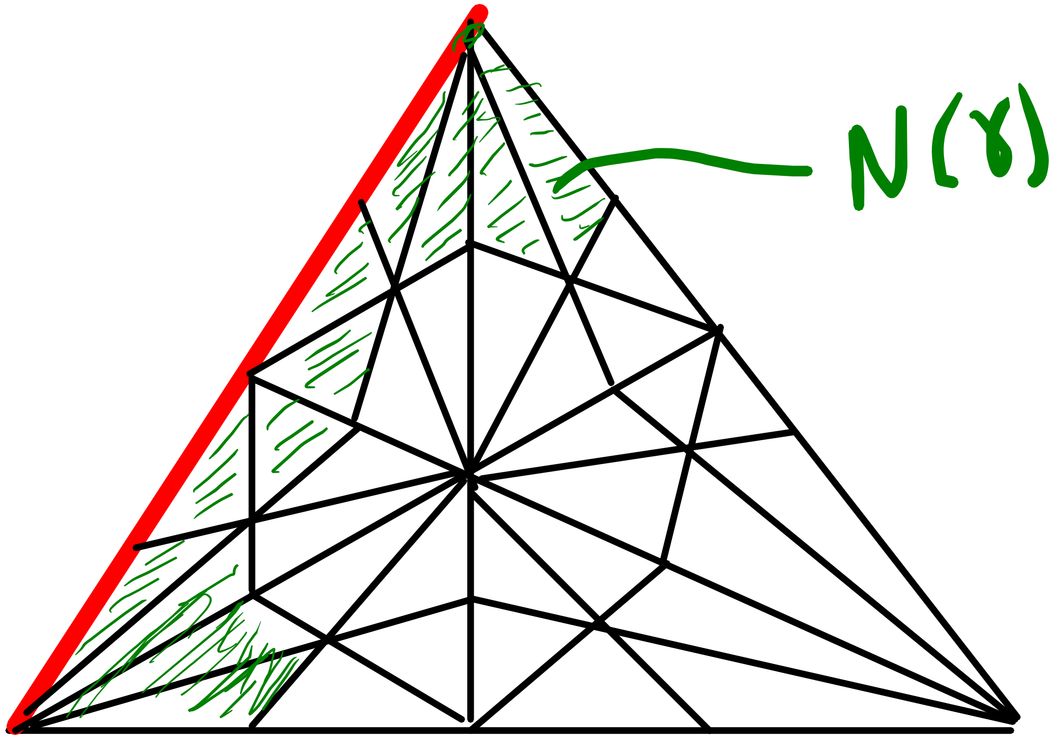




↖ $N(\gamma) =$ everything

Solution: Barycentric subdivision.

$N(\gamma) \subset K^2$ union of simplices in K^2 that meet γ.



(combinatorial) tubular neighborhood

Lemma (loop nbhd) $\gamma \subset K$ edge loop.

$N(\gamma) \subset K^2$ has $|N(\gamma)| \cong A$ or M.

Lemma (tree nbhd) $T \subset K$ tree, $N(T) \subset K^2$

$\Rightarrow |N(T)| \cong D^2$

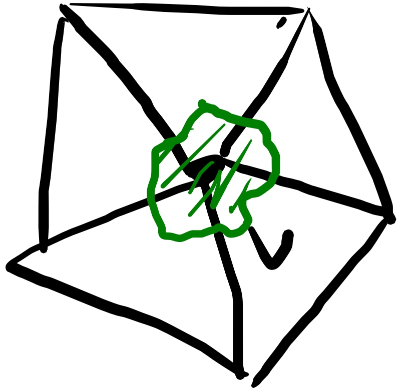
Lemma (tree nbhd) $T \subset K$ tree, $N(T) \subset K^2$

$\Rightarrow |N(T)| \cong D^2$

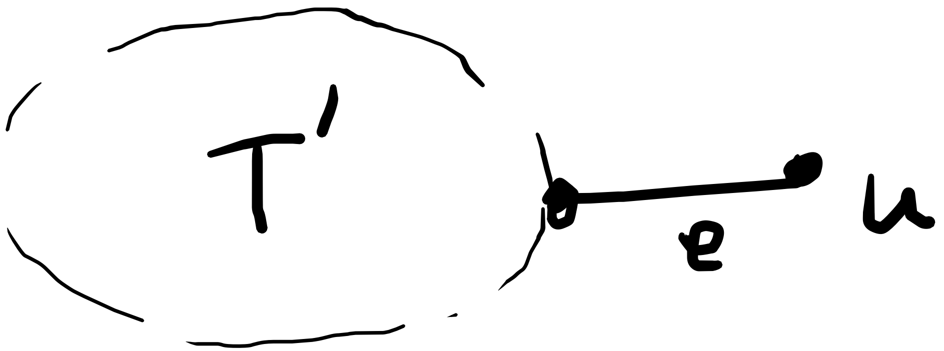
Proof By induction on # vertices of T

Base case 1 vertex

Draw/check this.

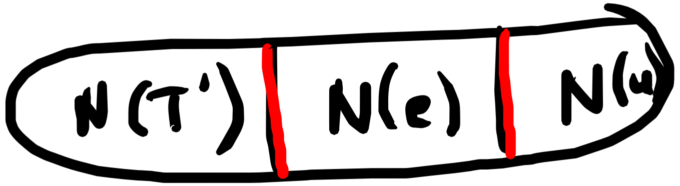


Induction step Take u vertex w/ valence 1.

$T =$ 

$$N(T) = N(T') \cup N(e) \cup N(u)$$
$$\cong D^2 \cup D^2 \cup D^2$$

<u>Check</u>: these are glued along arcs in ∂

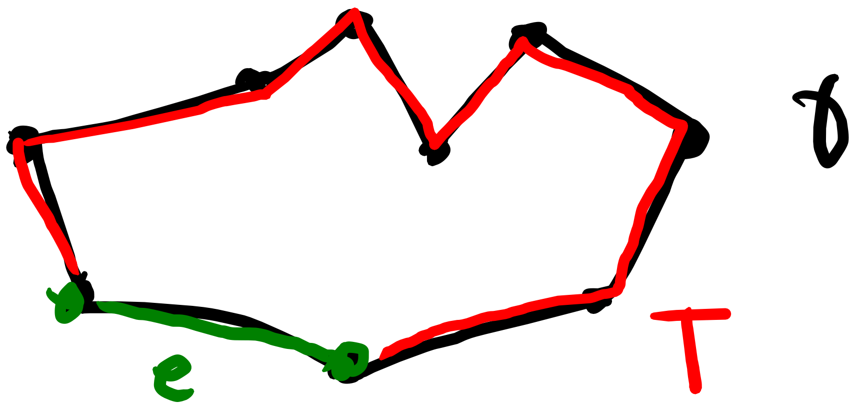


$\square$

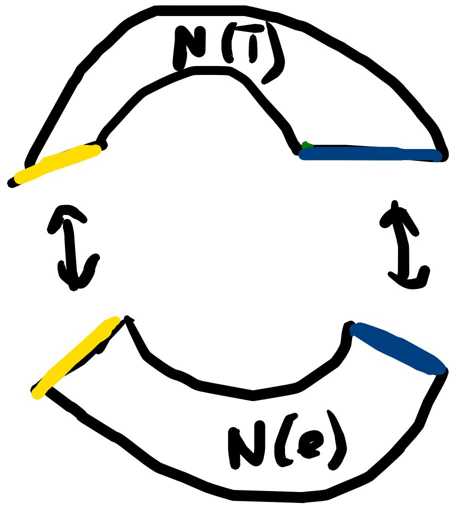
Lemma (loop nbhd) $\gamma \subset K$ edge loop.

$N(\gamma) \subset K^2$ has $|N(\gamma)| \cong A$ or M.

Proof.



$\gamma = e \cup T$



$$N(\gamma) = N(e) \cup N(T)$$

either get $|N(\gamma)| = A$ or M

(depending on gluing).

$\square$

Combinatorial Surgery

$|K|$ combinatorial surf., γ edge loop, $N(\gamma) \subset K^2$

$|N(\gamma)| \cong A$ or M.

Surgery combinatorial surface $|\hat{K}|$ obtained from $\underbrace{|K^2| \setminus \mathrm{int}|N(\gamma)|}_{K_0}$ by coning off each boundary component

Exercise

- if $|N(\gamma)| \cong A \Rightarrow \chi(\hat{K}) = \chi(K) + 2$
- " $\cong M \Rightarrow \chi(\hat{K}) = \chi(K) + 1$

use additivity $\chi(K_1 \cup K_2) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)$

$$\chi(K) = \chi(K_0) + \cancel{\chi(N(\gamma))}^{\to 0} - \cancel{\chi(\partial N(\gamma))}^{\to 0}$$

$$\chi(\hat{K}) = \chi(K_0) + \underbrace{\chi(D^2 \sqcup D^2)}_{= 2} - \cancel{\chi(S^1 \sqcup S^1)}$$

↑ Annulus case.