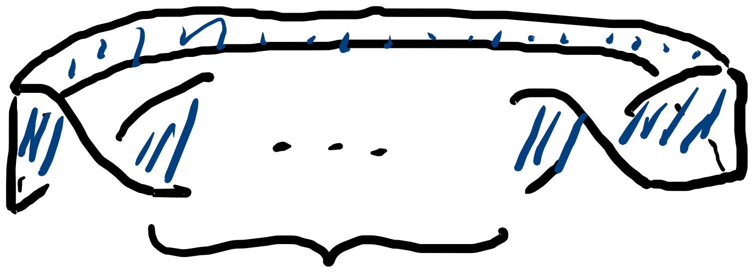
I. More topological equivalence

<u>Recall</u> $X \cong Y$ if $\exists$ continuous

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \quad \text{s.t.} \quad g \circ f = id_X, \quad f \circ g = id_Y$$

Say $f : X \longrightarrow Y$ is a <u>top. equivalence</u>.

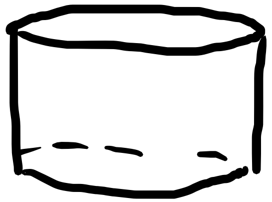
<u>Example</u> $X_n :=$



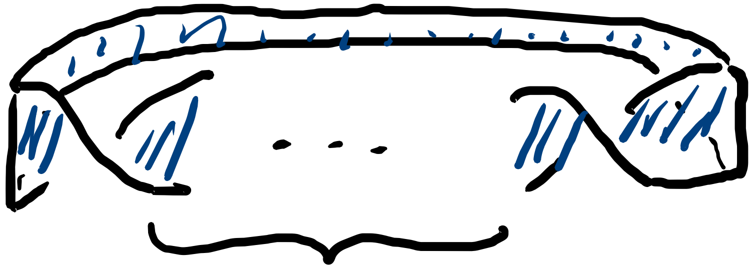
$n \geq 0$

n twists

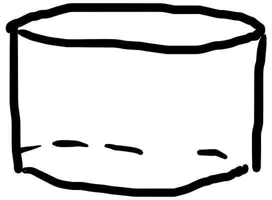
$X_0 =$



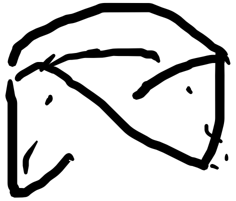
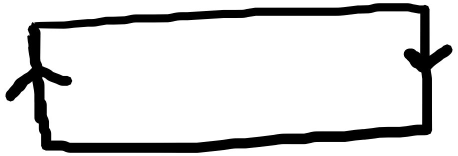
annulus $S^1 \times [0,1]$

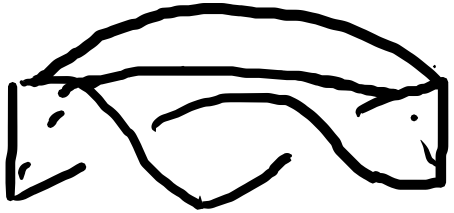
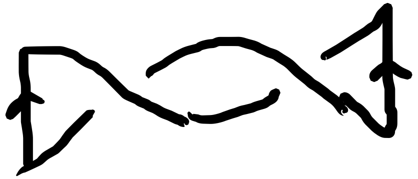
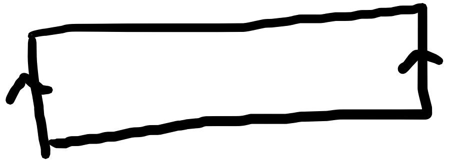
Example $X_n :=$  $n \geq 0$

n twists

$X_0 =$  annulus $S^1 \times [0,1] =$ 

no obvious bijection here

$X_1 =$  Möbius band $=$ 

$X_2 =$  $=$  $\longleftrightarrow$ 

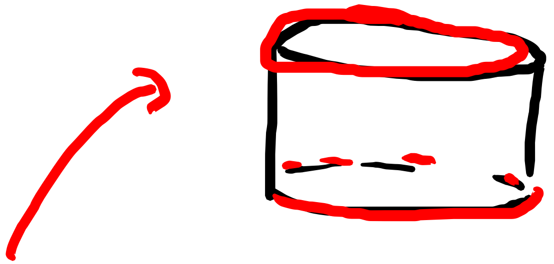
Claim $X_0 \cong X_2$

Pf: There is an obvious bijection

Similarly $\quad X_0 \cong X_{2k} \quad k \geq 1.$

and $\quad X_1 \cong X_{2k+1}$

OTOH, $\quad X_0 \not\cong X_1$: compare boundary



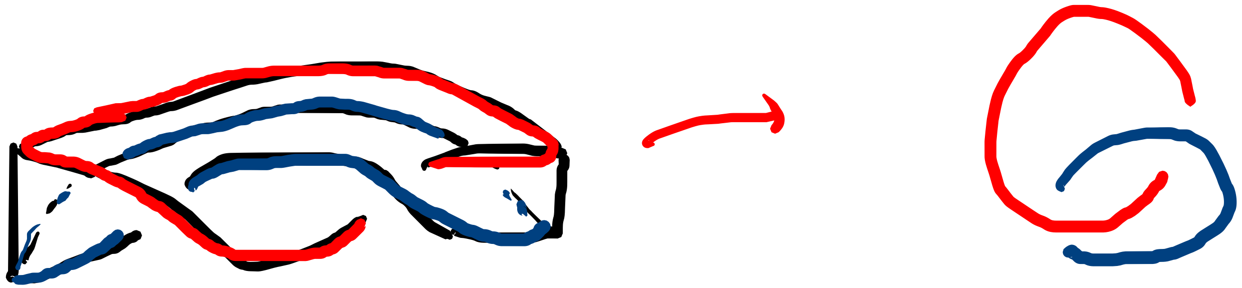
unlinked circles

(if $X_0 \cong X_1$ then $\partial X_0 \cong \partial X_1$ requires proof)

$\partial X_0 = S^1 \sqcup S^1 \not\cong S^1 = \partial X_1$

<u>Rmk</u> $\quad X_0, X_2$ cannot be deformed one to other in $\mathbb{R}^3$

(not "isotopic")

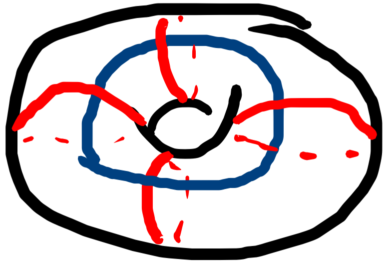


nontrivial link

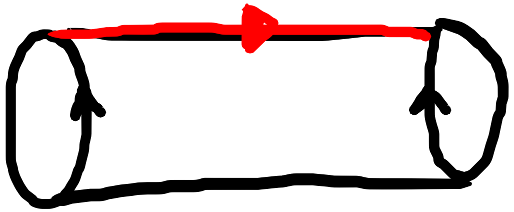
if can deform X_0 to X_2 in $\mathbb{R}^3$

then can deform OO to

Ex $T = \text{torus} := S^1 \times S^1$



$=$



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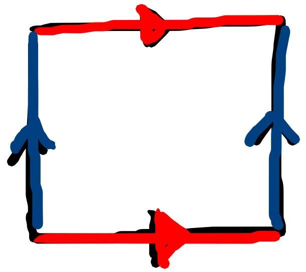

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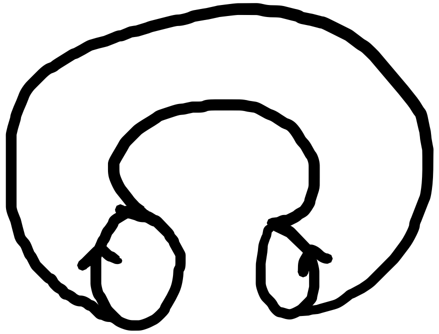
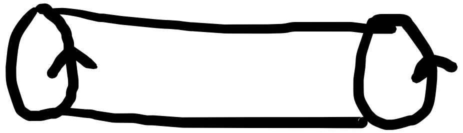
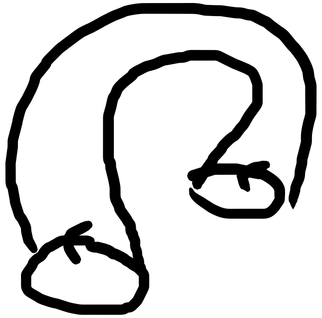
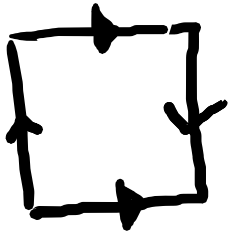


Alternatively we could consider



This gives a different space



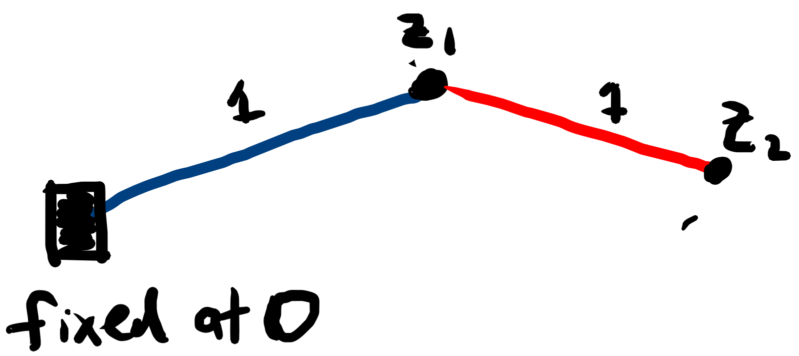


K = Klein bottle.

Thm $T \not\cong K$

Ex (linked rods)

consider linkage



$\subset \mathbb{R}^2$

consider the configuration space

$X = \{$ possible positions of linkage $\}$

$= \left\{ (z_1, z_2) : \begin{array}{l} z_1, z_2 \in \mathbb{R}^2 \\ |z_1| = 1 \quad |z_2 - z_1| = 1 \end{array} \right\}$

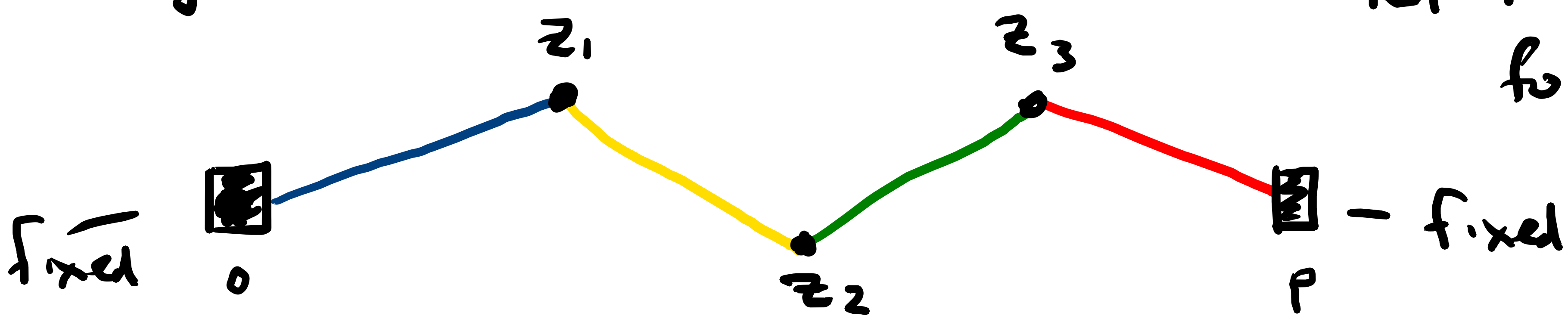
another linkage

$|z_1| = 1$

$|z_2 - z_1| = 1$

$|z_3 - z_2| = 1$

$|p - z_3| = 1$



let Y config. space for linkage

$X \cong Y$

dimension count: X is 2 dimensional

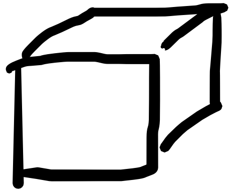
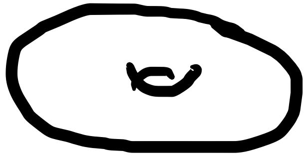
(HW2).

Y is also 2-dimensional.

<u>Basic problem in topology</u>: given X, Y spaces, determine if $X \cong Y$.

<u>Thm</u> P, Q polyhedra. If $P \cong Q$, then $\chi(P) = \chi(Q)$

Consequently if $\chi(P) \neq \chi(Q)$ then $P \not\cong Q$.

eg. $S^2 \cong$  $\not\cong$ torus polyhedron $\cong$ 

$\chi = 2$ $\chi = 0$

Basic problem in topology: given X, Y spaces, determine if $X \cong Y$.

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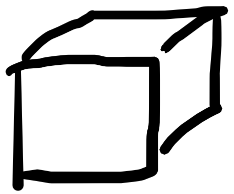
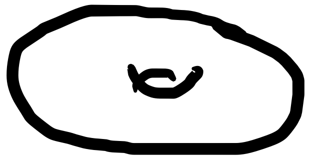
eg. $S^2 \cong$  $\not\cong$ torus polyhedron $\cong$ 

$\chi = 2$ $\qquad$ $\chi = 0$

Rmk We proved a special case: if P, Q satisfy assumptions of Euler's thm then $P \cong S^2 \cong Q$. Theorem predicts $\chi(P) = \chi(Q)$. Indeed by Euler's thm $\chi(P) = 2 = \chi(Q)$.

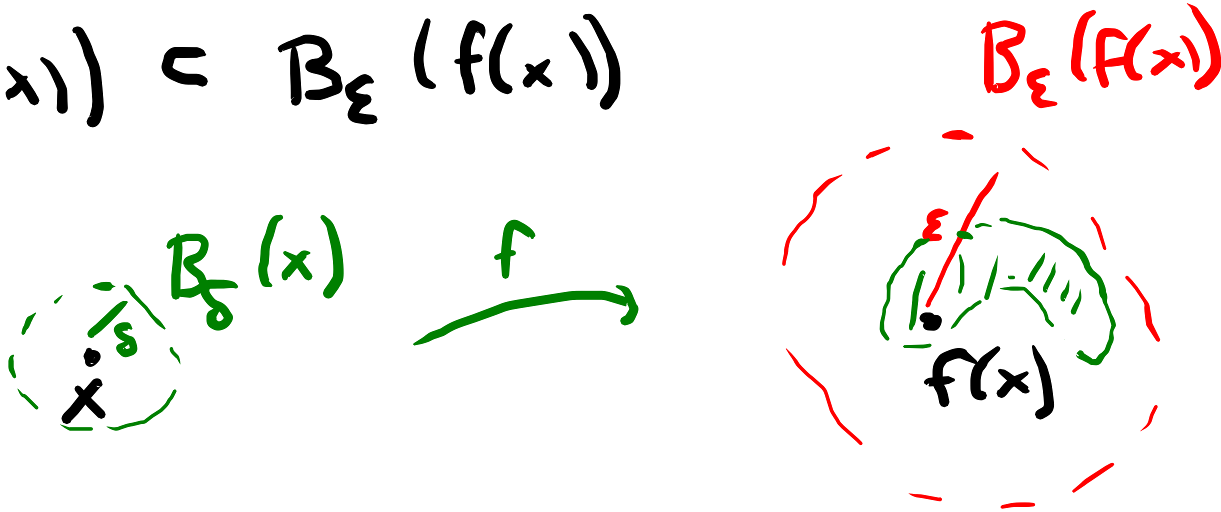
II. Topological Spaces

Want: general definition of space that includes all previous examples and more.

roughly "a space is a set w/ extra structure so that it makes sense to talk about continuous functions"

<u>Recall</u> $f : \mathbb{R}^n \to \mathbb{R}^n$ is <u>continuous</u> if for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$

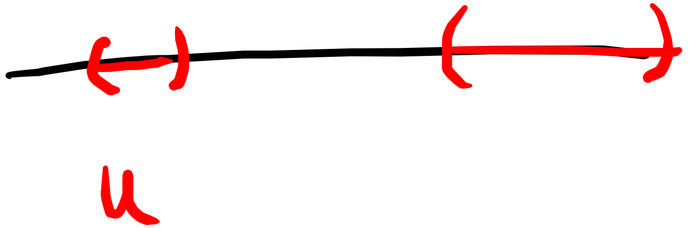
$B_\delta(x) = \{ y \in \mathbb{R}^n : |x - y| < \delta \}$



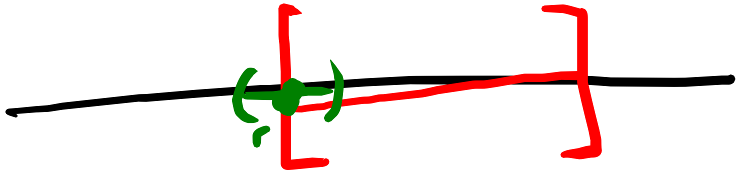
Say $U \subset \mathbb{R}^m$ <u>open</u> if for $y \in U$

$\exists\, r > 0$ s.t. $B_r(y) \subset U$.

<u>Eg</u> open subsets of $\mathbb{R}$ are unions of open intervals



closed intervals are not open



can restate: $f : \mathbb{R}^n \to \mathbb{R}^m$ <u>continuous</u> if

$f^{-1}(U)$ open for every open $U \subset \mathbb{R}^m$.

$$\overset{\shortparallel}{\{x \in \mathbb{R}^n \mid f(x) \in U\}}.$$

Basic properties of open sets in $\mathbb{R}^n$ motivates:

<u>Defn</u> A <u>topological space</u> is a set X together with a collection of subsets of X, called open sets, s.t.

(i) X, ϕ are open

(ii) (arbitrary) unions of open sets are open

(iii) finite intersections of open sets are open.

<u>Defn</u> X, Y top. spaces, $f: X \to Y$ <u>continuous</u> if for each $U \subset Y$ open $f^{-1}(U) \subset X$ also open.

Ex If $X \subset \mathbb{R}^3$ can make X a topological space by declaring sets of form $U \cap X$ are open when $U \subset \mathbb{R}^3$ open.

Exercise: this defines a topology on X.