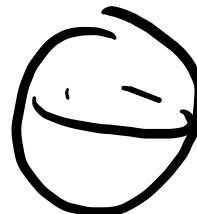


# I. Classification of Surfaces

Peth A closed Surface is a compact, connected, Hausdorff space  $S$  s.t.  $\forall x \in S \exists \epsilon \in U \subset S$  s.t.

$$U \cong R^2$$

Examples :



...

$$RP^2, K = \begin{array}{c} \square \\ \swarrow \searrow \end{array}$$

Nonexamples •  $\mathbb{R}^2$ ,  $T^2 \setminus \text{pt}$  (not compact)

•  $D^2$ ,  $M = \text{M\"obius}$ ,  $S^1 \times I$ . These are surfaces with boundary

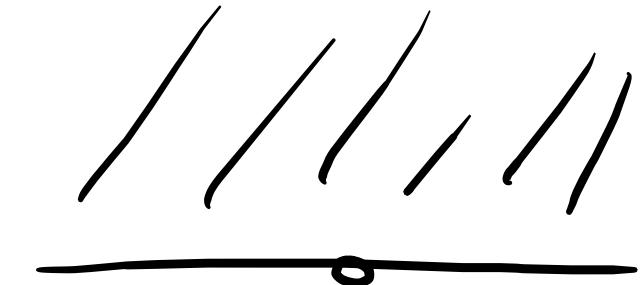


Points  $p$  on boundary have neighborhoods

$$U \cong \mathbb{R} \times [0, \infty) \neq \mathbb{R}^2.$$

Claim  $\mathbb{R}^2 \neq \mathbb{R} \times [0, \infty)$ .

Pf: Consider  $X = \mathbb{R} \setminus [0, \infty) \setminus \{(0, 0)\}$



Check:  $\pi_1(X) = \{1\}$ . OTOH  $\forall p \in \mathbb{R}^2 \quad \pi_1(\mathbb{R}^2 \setminus p) \cong \mathbb{Z}$ .

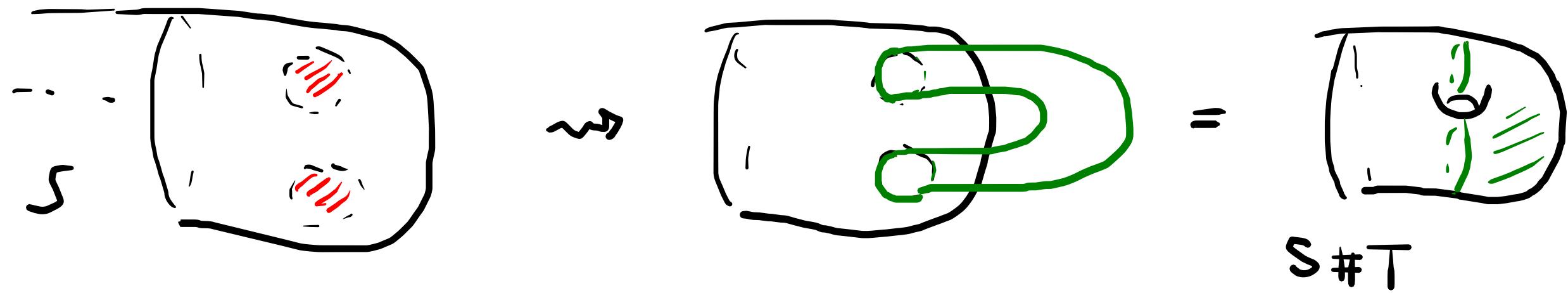
□

Problem Give a complete list of all closed surfaces. Do same for surfaces with boundary.

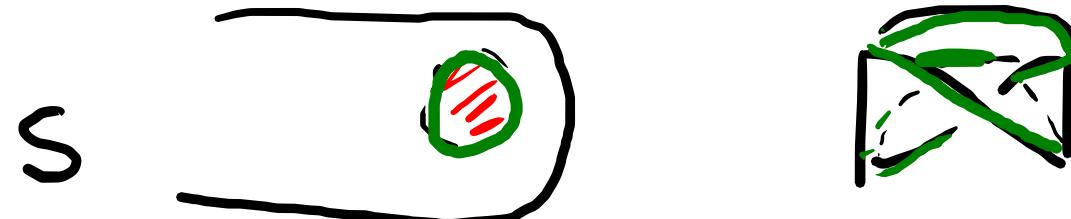
Rmk A compact, con., Hausdorff  $X \cong \mathbb{R}$  is locally  $S^1$ .

Surface operation Fix  $S$  surface.

- ① Annulus attachment : remove 2 <sup>open</sup> disk and  
glue on an annulus A

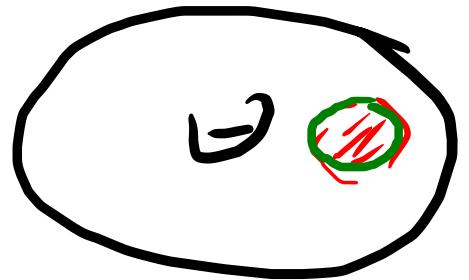


- ② Möbius attachment : remove disk and glue M

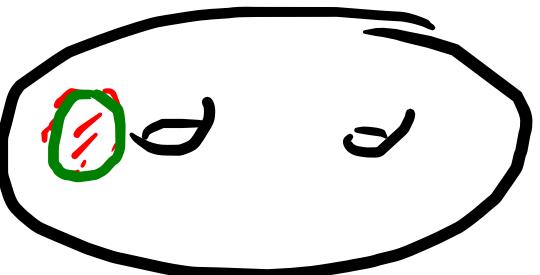


③ connected sum :

$S_1, S_2$  surfaces, remove disk from  
each and glue

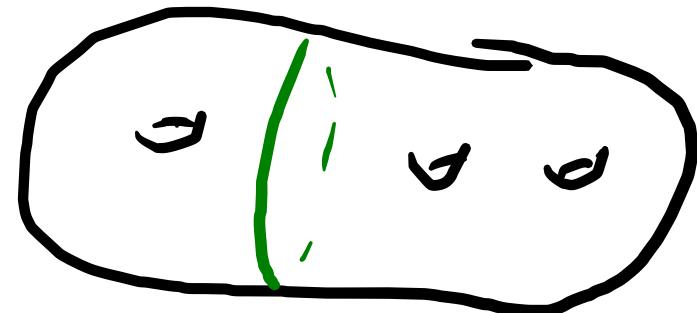


$S_1$



$S_2$

$\rightsquigarrow$

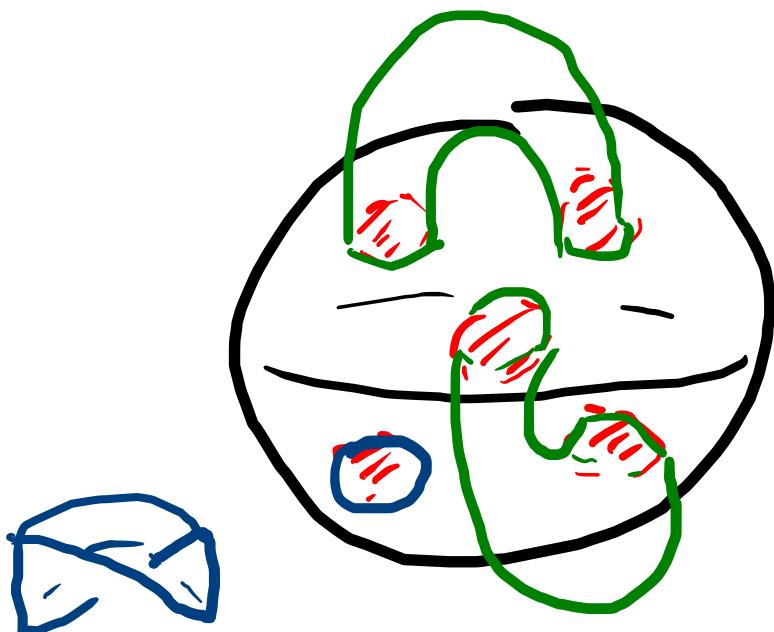


$S_1 \# S_2$

Theorem (classification of closed surfaces)

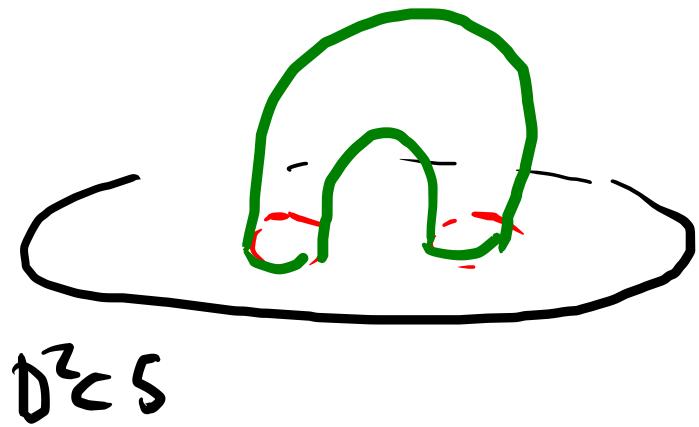
Every closed surface is obtained from  $S^2$  by applying operations (1), (2) finitely many times.

Rmk This list  
is redundant.

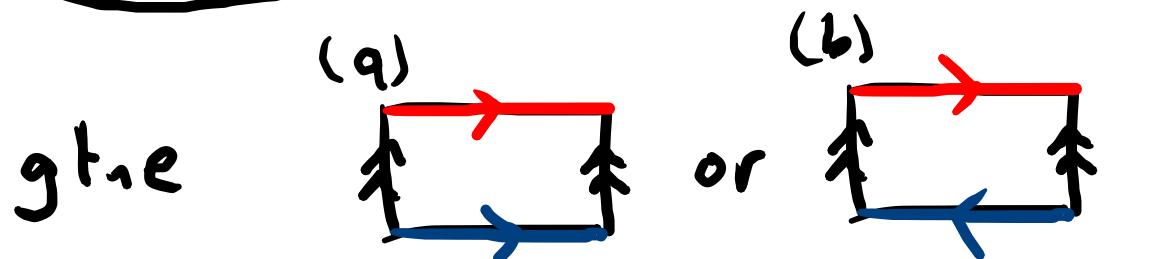
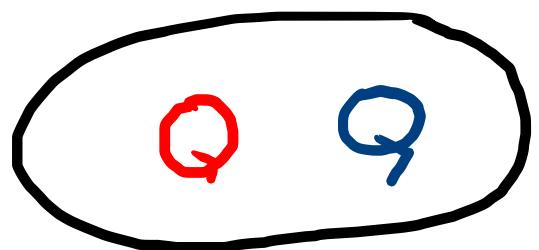
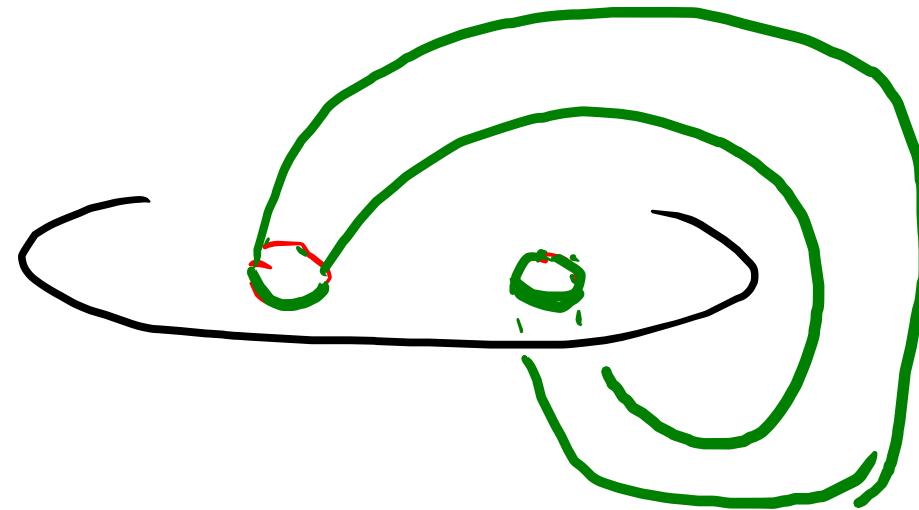


# Redundancy / well-definedness

- Annulus attachment is not well-defined:



$\partial^2 \subset S$



g.t.e

$$(a) \quad T^2 \setminus D^2 =$$

A diagram of a torus  $T^2$  represented by a rectangle with opposite edges identified. A blue horizontal line segment represents a disk  $D^2$  removed from the torus. A red arrow points along the boundary of the torus.

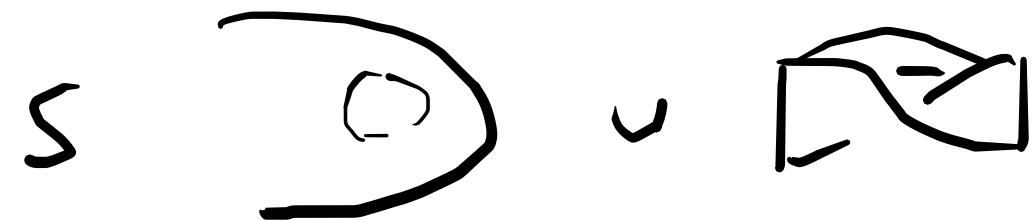
$$(b) \quad K \setminus D^2 =$$

A diagram of a surface  $K$  represented by a rectangle with opposite edges identified. A blue horizontal line segment represents a disk  $D^2$  removed from the surface. A red arrow points along the boundary of the surface.

Thus

Annulus attachment (a) is  $S \rightsquigarrow S \# T$   
(b) is  $S \rightsquigarrow S \# K$

- Möbius attachment as # operation.

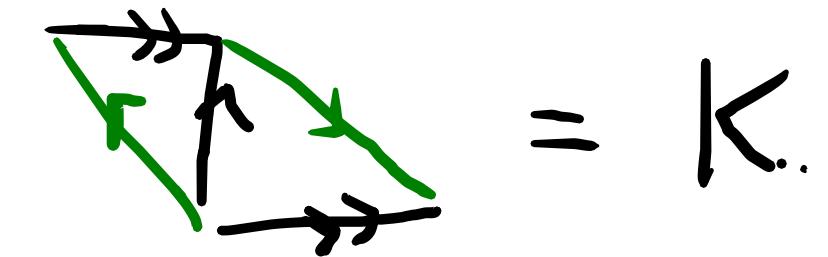
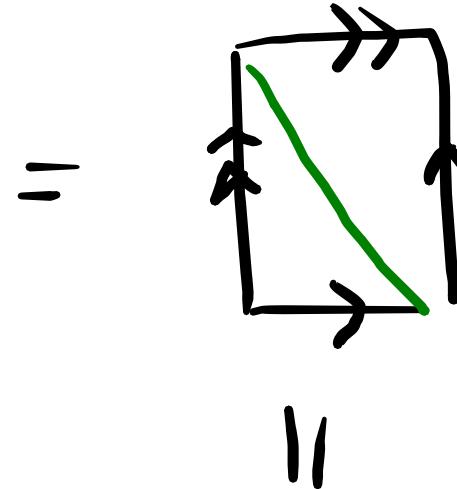
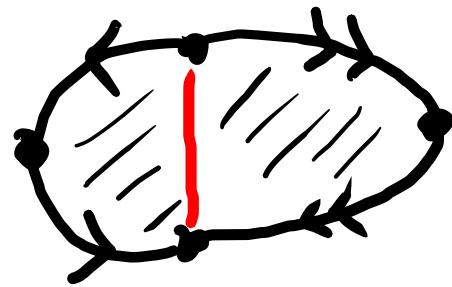
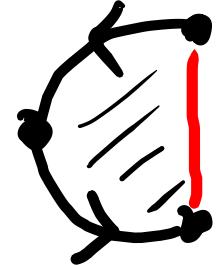
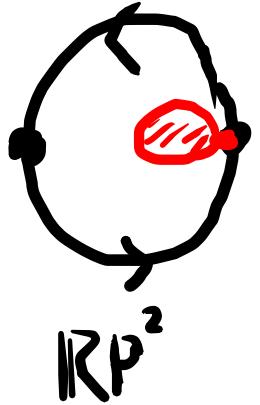


Möbius attachment

$$\text{is } S \rightsquigarrow S \# RP^2$$

Recall  $RP^2 = M \cup D^2$

- $\mathbb{R}P^2 \# \mathbb{R}P^2 = K$

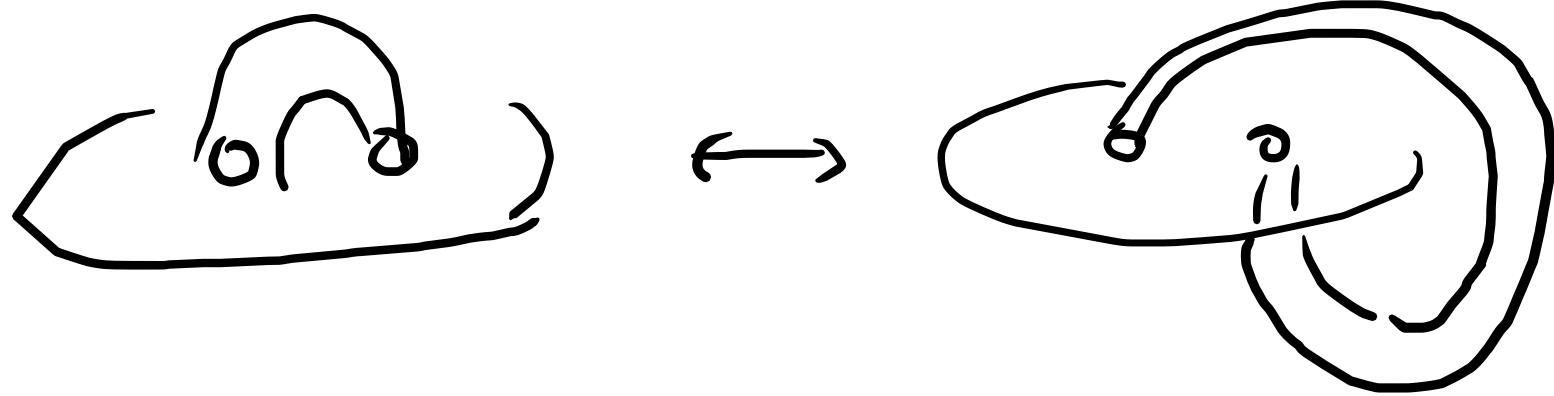


Consequently

$$S \xrightarrow{\quad} S \# K \quad \text{Same as}$$

$$S \xrightarrow{\quad} S \# (\mathbb{R}P^2 \# \mathbb{R}P^2)$$

$$\mathbb{R}P^2 \# T^2 \cong \mathbb{R}P^2 \# K$$



$\mathbb{R}P^2$  is 1-sided ("non-orientable")

"  
 $M \cup D^2$



Or For  $n \geq 1$ .

$$\begin{aligned} (\mathbb{T}^2)^{\#m} \# (\mathbb{RP}^2)^n &\simeq (\mathbb{K})^{\#m} \# (\mathbb{RP}^2)^{\#n} \\ &\simeq (\mathbb{RP}^2)^{\#2m+n}. \end{aligned}$$

---

Theorem (improved classification of surfaces) Every closed surface is either  $S^2$ ,  $(\mathbb{T}^2)^{\#m}$ , or  $(\mathbb{RP}^2)^{\#k}$  and no two of these are  $\cong$ .

$$\underline{\text{Rank}} \quad S^2 \# T^2 \stackrel{\sim}{=} T^2$$

$$\text{In general} \quad S^2 \# S \stackrel{\sim}{=} S.$$