I. Proof of van Kampen

$K = K_1 \cup K_2$, $p \in K_1 \cap K_2$

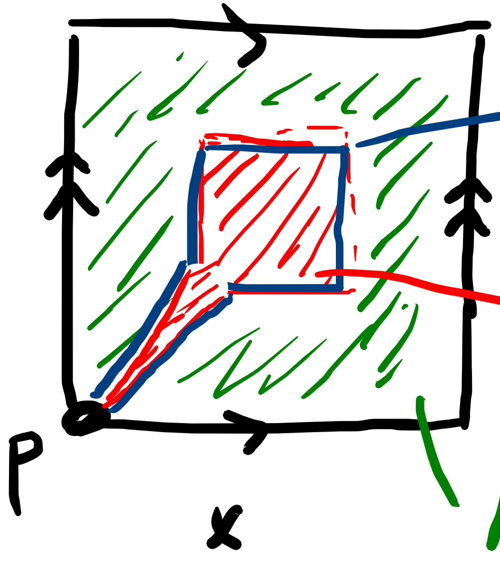
$$\begin{tikzcd}
 & E(K_1, p) \\
E(K_1 \cap K_2, p) \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & \\
 & E(K_2, p)
\end{tikzcd}$$

(van Kampen) If $K_1 \cap K_2$ connected then

$$E(K, p) \cong E(\underline{K_1, p}) * E(\underline{K_2, p}) \Big/ \langle\!\langle j_1(\varepsilon) j_2(\varepsilon)^{-1} : \varepsilon \in E(K_1 \cap K_2) \rangle\!\rangle$$

Example T^2

$$\pi_1(T^2) = \langle x, y \mid xyx^{-1}y^{-1} = 1 \rangle \cong \mathbb{Z}^2$$



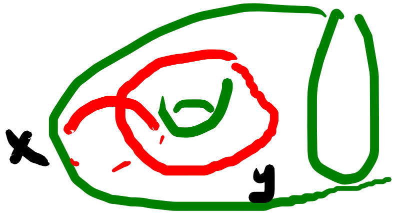
$A \cap B \cong S^1$

$B \cong D^2$

$\pi_1(A \cap B) = \mathbb{Z} = \langle \varepsilon \rangle$

$\pi_1(A) = \mathbb{Z} * \mathbb{Z} = \langle x, y \rangle$

$A \cong T^2 \setminus D^2 \cong S^1 \vee S^1$



$$\pi_1(T^2) = \underbrace{\pi_1(A)}_{\langle x, y \rangle} * \cancel{\pi_1(B)} \Big/ \langle\langle xyx^{-1}y^{-1} \rangle\rangle$$

$$\pi_1(A \cap B) \xrightarrow{j_1} \pi_1(A) \qquad \pi_1(A \cap B) \xrightarrow{j_2} \pi_1(B)$$

$$\varepsilon \longmapsto xyx^{-1}y^{-1} \qquad \varepsilon \longmapsto 1.$$

Recall presentation for $E(L,p)$:

Choose $T \subset L$ max tree

Write $v_1, \ldots, v_d$ for vertices of L

$$E(L,p) \cong \left\langle g_{ij} \text{ for } \{v_i, v_j\} \in L \;\middle|\; \begin{array}{l} g_{ij} = 1 \text{ if } \{v_i, v_j\} \in T \\ g_{ij} g_{jk} = g_{ik} \text{ if } \\ \qquad \{v_i, v_j, v_k\} \in L \end{array} \right\rangle$$

Proof of van Kampen

Compare presentations for $E(L, p)$

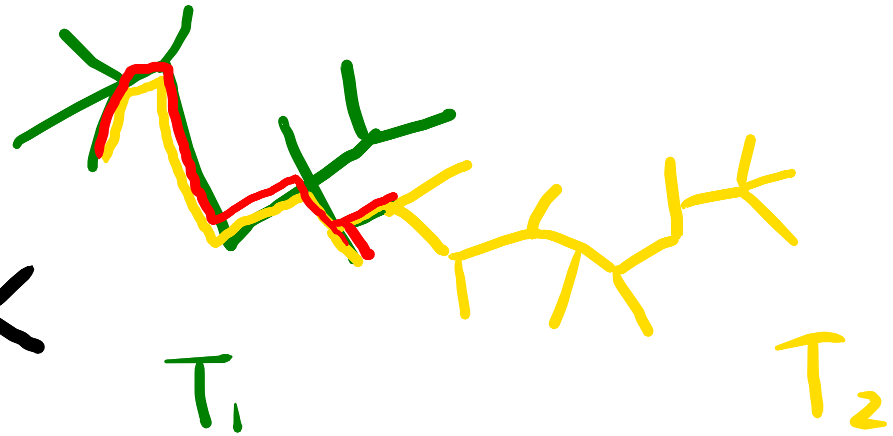
w/ $L = K_0, K_1, K_2, K_1 \cap K_2$.

Choose max tree $T_{12} \subset K_1 \cap K_2$

Extend to max tree $T_1 \subset K_1$ & $T_2 \subset K_2$

Set $T = T_1 \cup T_2$.

Observe T is a max tree of K



$$E(K_1) = \left\langle a_{ij} \text{ for } \{v_i,v_j\}\in K_1 \;\middle|\; \begin{array}{ll} a_{ij}=1 & \{v_i,v_j\}\in T_1 \\ a_{ij}a_{jk}=a_{ik} & \{v_i,v_j,v_k\}\in K_1 \end{array} \right.$$

$$*$$

$$E(K_2) = \left. b_{ij} \text{ for } \{v_i,v_j\}\in K_2 \;\middle|\; \begin{array}{ll} b_{ij}=1 & \{v_i,v_j\}\in T_2 \\ b_{ij}b_{jk}=b_{ik} & \{v_i,v_j,v_k\}\in K_2 \end{array} \right\rangle$$

$$E(K) = \left\langle g_{ij} \text{ for } \{v_i,v_j\}\in K \;\middle|\; \begin{array}{ll} g_{ij}=1 & \{v_i,v_j\}\in T \\ g_{ij}g_{jk}=g_{ik} & \{v_i,v_j,v_k\}\in K \end{array} \right\rangle$$

$\{g_{ij}\} = \{a_{ij}\} \cup \{b_{ij}\}$ but this is not nec. a disjoint union

Compare presentations for $E(K_1) * E(K_2)$ and $E(K)$.

To get from presentation for $E(K_1) * E(K_2)$ to presentation for $E(K)$ need only add relations $a_{ij} = b_{ij}$ if $\{v_i, v_j\} \in K_1 \cap K_2$.

$\square$

II. Simplicial approximation

<u>Recall</u> $s: |K| \longrightarrow |L|$ is <u>simplicial</u> if sends simplices to simplices linearly.

For $x \in |K|$ the <u>carrier</u> of x $\operatorname{carr}(x)$ is the unique simplex $\sigma \in K$ s.t. $x \in \operatorname{int}|\sigma|$.

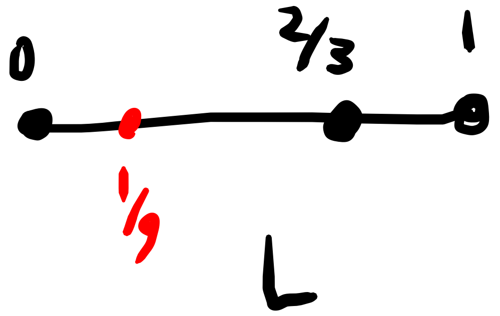
$f, s: |K| \longrightarrow |L|$, s simplicial, Say s is a <u>simplicial approximation</u> of f if $s(x) \in \operatorname{carr}(f(x))$ $\forall x \in |K|$.

Ex $|K| = [0,1]$

$f:[0,1] \longrightarrow [0,1]$ $\quad f(x) = x^2$

Then $s(x) = \underline{x}$ is a simplicial approx.

Ex $|K| = |L| = [0,1]$



$f(x) = x^2$ properties of a simplicial approximation s

if $f(x)$ is vertex of L then $s(x) = f(x)$

$\Rightarrow s(0) = 0, \; s(1) = 1, \; s\left(\sqrt{\tfrac{2}{3}}\right) = \tfrac{2}{3}$

s simplicial so $s(\frac{1}{3})$ is a vertex of L

$s(\frac{1}{3}) \in \mathrm{carr}(f(\frac{1}{3})) = [0, 2/3] \Rightarrow s(\frac{1}{3}) \in \{0, \frac{2}{3}\}$

$\quad\quad\quad\quad\quad\quad 1/4$

<u>Case</u> $s(\frac{1}{3}) = 0$. Then $s([\frac{1}{3}, 1]) = [0, 1]$

$s(x) = \frac{3}{2}x - \frac{1}{2} \Rightarrow s(\sqrt{\frac{2}{3}}) \neq \frac{2}{3}$

Similarly if $s(\frac{1}{3}) = \frac{2}{3}$ find contradiction.

$\Rightarrow f$ does not have a simplicial approximation

Lemma If s is a simplicial approx of $f : |K| \longrightarrow |L|$ then $f \sim s$.

Proof. WTS $\exists$ $h : |K| \times [0,1] \longrightarrow |L|$ s.t.

$$h|_{|K| \times 0} = f \qquad h|_{|K| \times 1} = s.$$

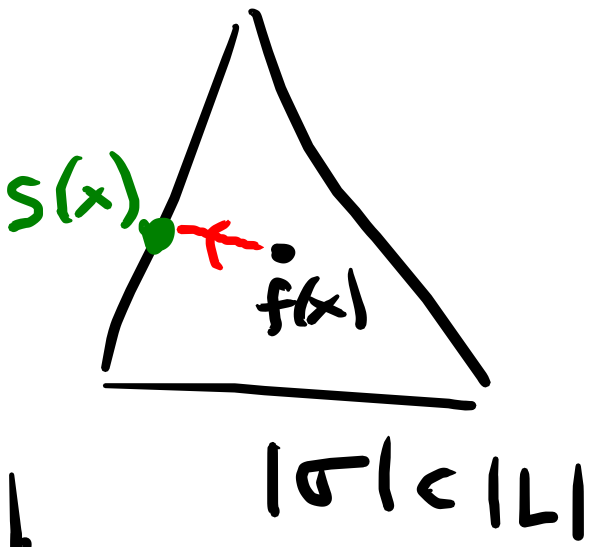
recall $|K|, |L| \subset \mathbb{R}^N$

Consider $h(x,t) = (1-t)\, f(x) + t \cdot s(x)$.

Need to check $h(x,t) \in |L|$ $\forall$ x,t.

For $x \in |K|$, let $\sigma = \text{carr}(f(x))$.

Know $s(x) \in |\sigma|$



$|\sigma|$ is convex so straight line from $f(x)$ to $s(x)$ is contained in $|\sigma|$.

$\square$

Thm (simplicial approximation)

$f: |K| \longrightarrow |L|$ any map. $\exists$ subdivision K' of K s.t. $f: |K'| = |K| \longrightarrow |L|$ has a simplicial approximation.