I. Van Kampen's Theorem

<u>Recap</u> X space, compute $\pi_1(X)$

if $X = |K|$, then $\pi_1(X) \cong \pi_1(|K|) \cong E(K,p) \cong G(K,T)$

edge group theorem

$G(K,T)$ is straightforward to compute, although possibly tedious.

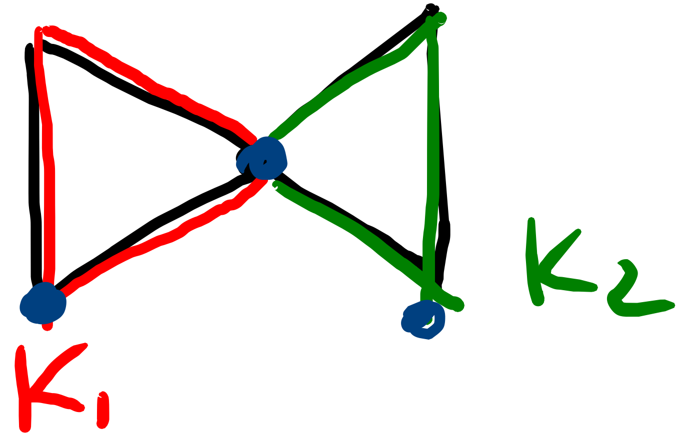
<u>Today</u> $X = A \cup B$. Describe $\pi_1(X)$ in terms of $\pi_1(A)$, $\pi_1(B)$, $\pi_1(A \cap B)$.

K simplicial complex

K_1, K_2 subcomplexes w/ $K = K_1 \cup K_2$

$p \in K_1 \cap K_2$ vertex. Assume $K_1 \cap K_2$ connected

<u>Example</u> $|K| = S^1 \vee S^1$



If $L \subset K$ subcomplex, then

there's a homomorphism $E(L, p) \longrightarrow E(K, p)$

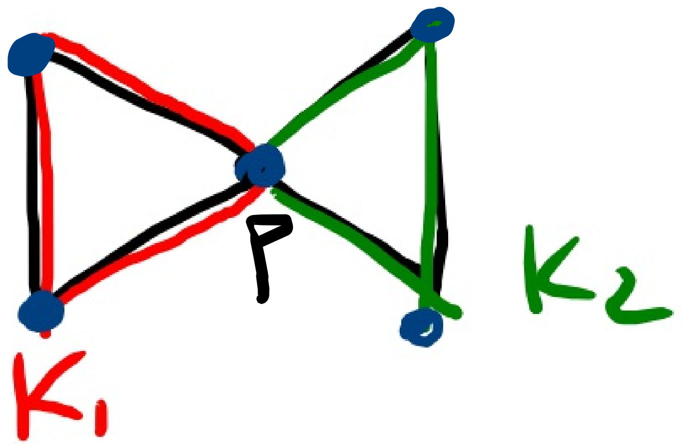
with $K = K_1 \cup K_2$

$$\begin{tikzcd} & E(K_1, p) \arrow[dr] & \\ E(K_1 \cap K_2, p) \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & E(K, p) \\ & E(K_2, p) \arrow[ur] & \end{tikzcd}$$

<u>Thm</u> (van Kampen) $E(K, p)$ is the quotient of $E(K_1, p) * E(K_2, p)$ by adding relations $j_1(\varepsilon) = j_2(\varepsilon)$ for $\varepsilon \in E(K_1 \cap K_2, p)$.

Examples

① $S^1 \vee S^1$

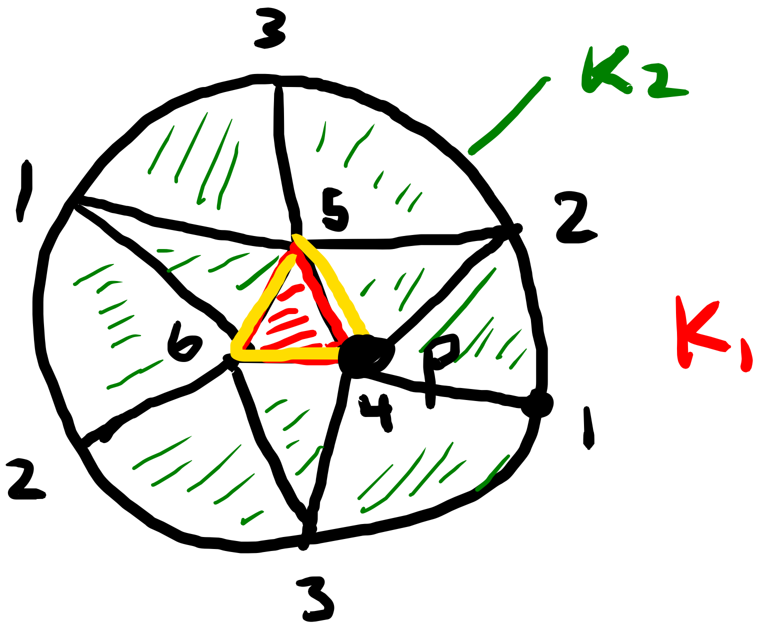


$K_1 \cap K_2 = \{p\}$

$$E(K,p) \cong E(K_1,p) * E(K_2,p)$$

$$\cong \mathbb{Z} * \mathbb{Z}$$

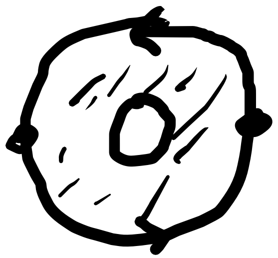
② $\mathbb{RP}^2$



$E(K_1,p) = \{1\}$

$E(K_2,p) \cong \mathbb{Z}.$

$E(K_1 \cap K_2, p) \cong \mathbb{Z}$



= Mobius

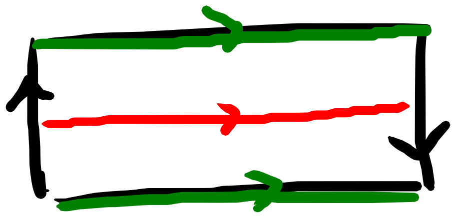
$E(K_1 \cap K_2, p)$ generated by 4564

$E(K_2, p)$ generated by 4514.

$$j_2 : \overset{\mathbb{Z}}{E(K_1 \cap K_2)} \xrightarrow{2} \overset{\mathbb{Z}}{E(K_2)}$$

$$4564 \longmapsto 4564 \sim (4514) * (4514).$$

$$E(K) \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$



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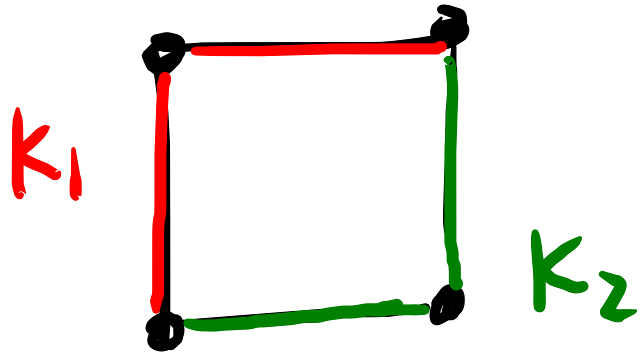
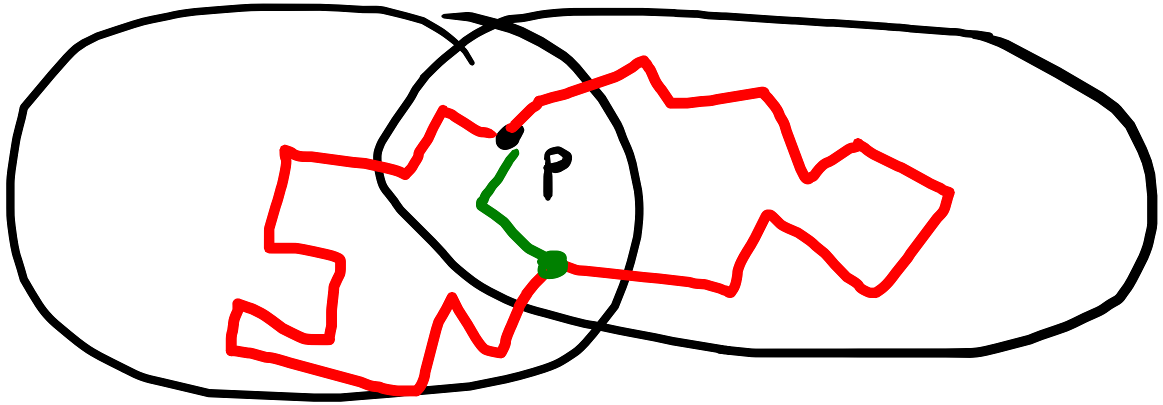


Van Kampen does not apply b/c $K_1 \cap K_2$ is not connected.

Intuition behind van Kampen

There is clearly a surjection

$E(K_1) * E(K_2) \to E(K)$:

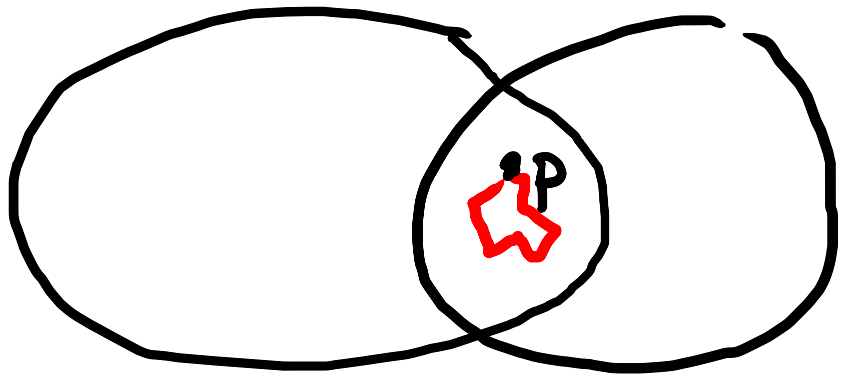


K_1 K_2

any edge path $pu_1 \cdots u_n p$ in K can be decomposed into finite concatenation of edge loop in K_1, K_2

There are some obvious elements of $\ker(E(K_1) * E(K_2) \longrightarrow E(K))$:

For $\varepsilon \in E(K_1 \cap K_2)$



ε can be viewed as

$\varepsilon_1 \in E(K_1)$ or $\varepsilon_2 \in E(K_2)$.

But $\varepsilon_1, \varepsilon_2$ map to same loop in $E(K)$

so $\varepsilon_1 \varepsilon_2^{-1} \in \ker$. Thm says these elements generate kernel.

II. Simplicial approximation

Edge group thm $\pi_1(|K|, p) \cong E(K, p)$

Defn K, L simplicial complexes. a map $f : |K| \longrightarrow |L|$ is simplicial if it sends simplices to simplices in a linear way.

Example

$|K| = [0,1]$ 

consider maps $|K| \longrightarrow |K|$

$$\left.\begin{array}{l} f(x) = x \\ \\ f(x) = 0 \end{array}\right\} \text{simplicial}$$

$$\left.\begin{array}{l} f(x) = 1/2 \\ \\ f(x) = x^2 \end{array}\right\} \text{not simplicial}$$

Given $f : |K| \longrightarrow |L|$ simplicial

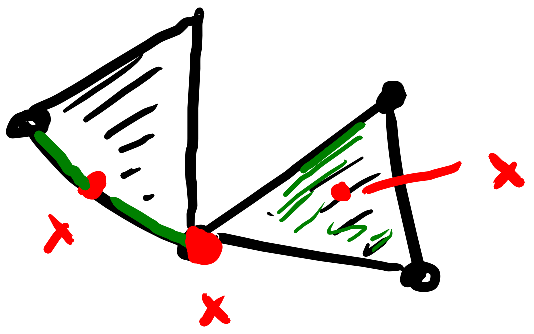
get $g : V(K) \longrightarrow V(L)$

and g completely determines f:

Given $x \in |K|$, $\exists$ unique simplex $\sigma \in K$

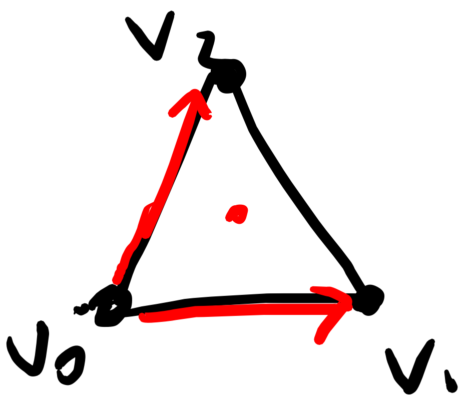
s.t. x is in the interior of $|\sigma|$

(σ is called the <u>carrier</u> of x)



$|\sigma| \subset |K| \subset \mathbb{R}^N$

$\llcorner$ simplex spanned $v_0, \dots, v_d$



$$x = \sum_{i=1}^{d} a_i (v_i - v_0)$$

$$f(x) = \sum a_i (f(v_i) - f(v_0)) \quad (f \text{ linear on } \sigma)$$

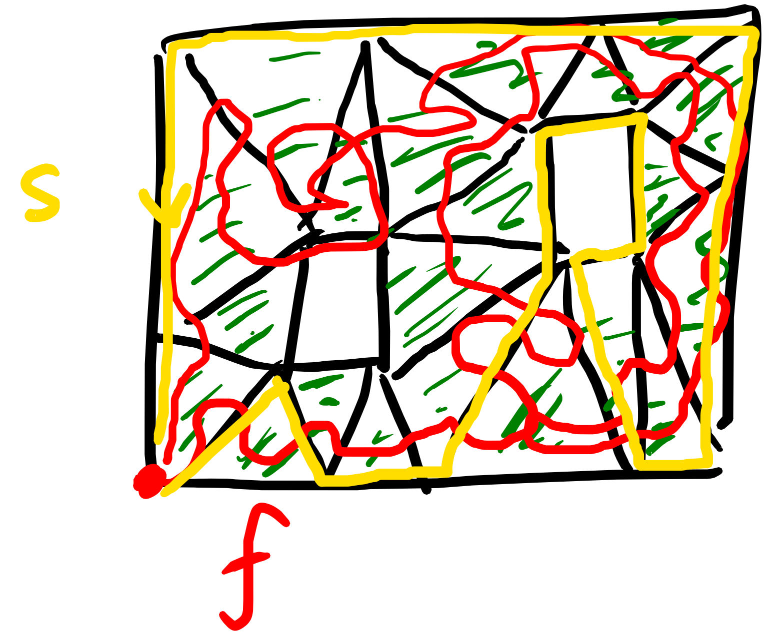
$$= \sum a_i (g(v_i) - g(v_0))$$

Note: given $g : V(K) \longrightarrow V(L)$ $\exists!$ simplicial $f : |K| \longrightarrow |L|$ s.t. $f|_{V(K)} = g$.

given any $f: |K| \longrightarrow |L|$

we'd like simplicial $s: |K| \longrightarrow |L|$

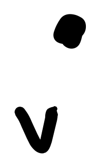
so that $f \sim s$.



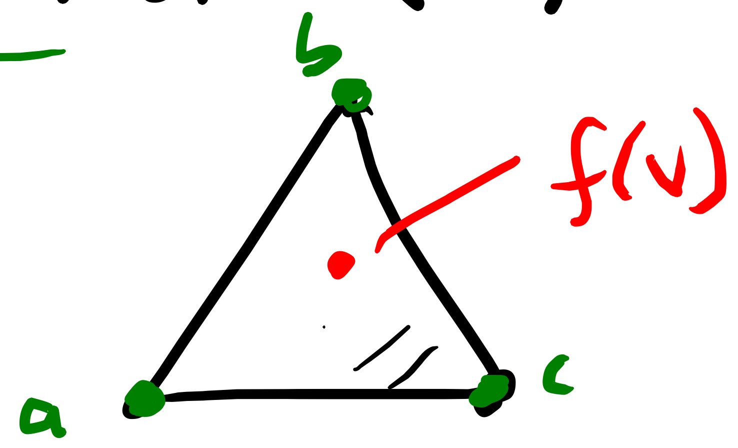
Defn. $f : |K| \to |L|$ any map.

Say simplicial map $s : |K| \to |L|$ is a <u>simplicial approximation</u> of f if

$s(x)$ is in <u>carrier</u> of $f(x)$ $\forall x \in |K|$.



vertex of K



triangle in $|L|$

a simplicial approx must send v to one of a, b, c.