# I. Presenting edge groups

$$E(K,p) \cong G(K,T)$$

$K$ simplicial complex, vertices $v_0, \dots, v_N$, $p = v_0$ basepoint

$T \subset K$ max tree

$G(K,T)$ generators : $g_{ij}$ whenever $\{v_i, v_j\} \in K$

relations : $g_{ij} = 1$ if $\{v_i, v_j\} \in T$

$g_{ij}\, g_{jk} = g_{ik}$ if $\{v_i, v_j, v_k\} \in K$

Rmk $G(K,T)$ has more gens & relations than last time but defines same group (exercise).

$g_{ij}, g_{ji}, g_{ii}$ | $g_{ii} = 1$, $g_{ij}\, g_{ji} = g_{ii} = 1$

$\Rightarrow g_{ji} = g_{ij}^{-1}$

**Thm** $G(K,T) \cong E(K,p)$

recall: $E(K,p)$ = edge paths $v_0 v_{i_1} \cdots v_{i_n} v_0$

up to equivalence

$uu \longleftrightarrow u$

$uvu \longleftrightarrow u$

$uvw \longleftrightarrow uw$ if 

$\in K$

**Proof of Thm** (proof is formal / algebraic)

We'll define $E(K,p) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} G(K,T)$ and show $\Phi \circ \Psi = id$

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<u>Define</u> $\Phi : E(K,p) \longrightarrow G(K,T)$

$\Phi(v_0 v_{i_1} \cdots v_{i_n} v_0) = g_{0,i_1}\, g_{i_1,i_2} \cdots g_{i_n,0}$

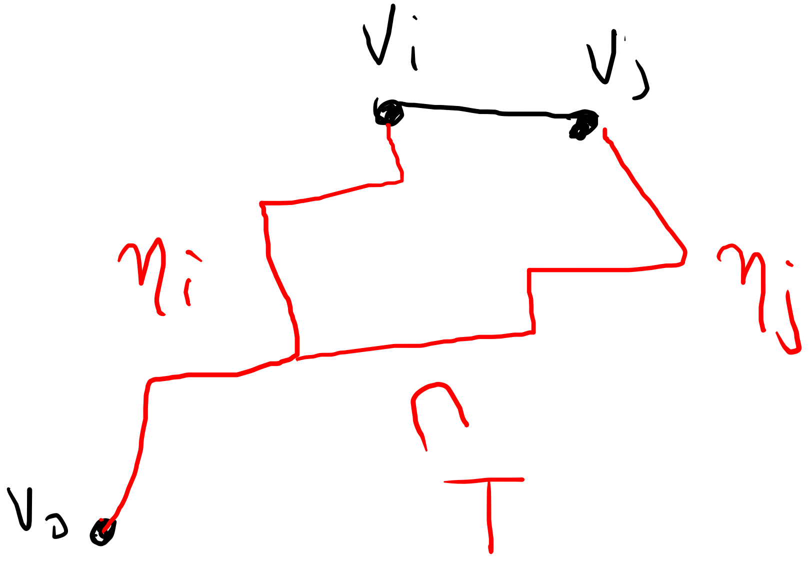
Note: $\Phi$ is a homomorphisms ✓

<u>Define</u> $\Psi : G(K,T) \longrightarrow E(K,p)$ by defining it on generators

For each $v_i \in V$ choose edge path $\eta_i$ from $v_0$ to $v_i$ in $T$. Denote $\overline{\eta}_i$ the reverse path $v_i$ to $v_0$.

Also choose $\eta_0 = v_0$ ("constant" path).

$\Psi(g_{ij}) = \eta_i v_i v_j \overline{\eta}_j$

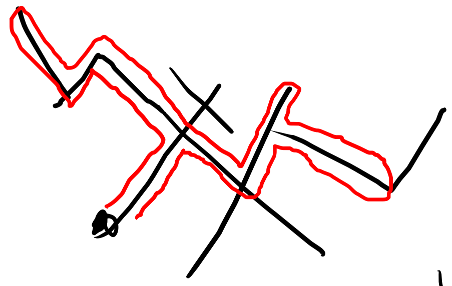



$$\Psi(g_{ij}) = \underbrace{\eta_i v_i v_j \bar{\eta}_j}_{\text{edge loop based at } p} \in E(K,p)$$

To show $\Psi$ is a hom. need to show that $\Psi$ send relations in $G(K,T)$ to relations in $E(K,p)$.

- if $\{v_i, v_j\} \in T$ $\quad g_{ij} = 1$

$\Psi(g_{ij}) = \eta_i v_i v_j \bar{\eta}_j$ edge loop in $T$



$T$ = tree $\Rightarrow$ this backtracks $\Rightarrow$ $\eta_i v_i v_j \bar{\eta}_j \sim v_0$.

$\Psi(g_{ij}) = [v_0]$

- if $\{v_i, v_j, v_k\} \in K$ want

$$\Psi(g_{ij} g_{jk}) = \Psi(g_{ik})$$

check:

$$\Psi(g_{ij} g_{jk}) = \Psi(g_{ij})\,\Psi(g_{jk})$$

$$= \eta_i v_i v_j \bar{\eta}_j \; \eta_j v_j v_k \bar{\eta}_k$$

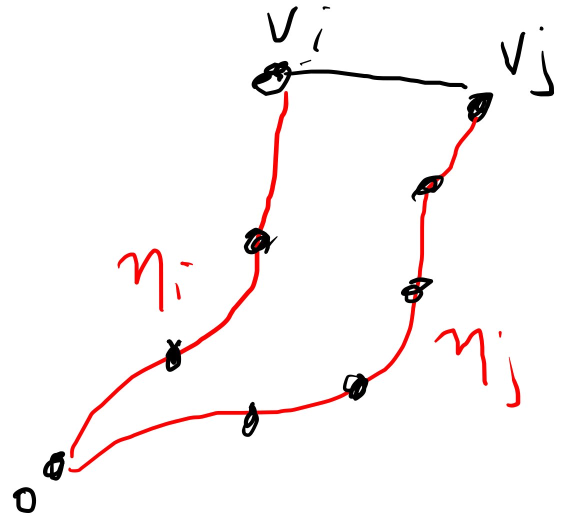
$$\sim \eta_i v_i v_j v_k \bar{\eta}_k$$

$$\sim \eta_i v_i v_k \bar{\eta}_k = \Psi(g_{ik}). \quad \checkmark$$

$\Phi, \Psi$ inverses

$$G(K,T) \xrightarrow{\Psi} E(K,p) \xrightarrow{\Phi} G(K,T)$$

$$\Phi(\Psi(g_{ij})) = \Phi(\eta_i v_i v_j \bar{\eta}_j) = g_{ij}$$




$$E(K,p) \xrightarrow{\Phi} G(K,T) \xrightarrow{\Psi} E(K,p)$$

$$\Psi(\Phi(v_0 v_{i_1} \cdots v_{i_n} v_0)) = \Psi(g_{0,i_1}\, g_{i_1,i_2} \cdots g_{i_n,0})$$

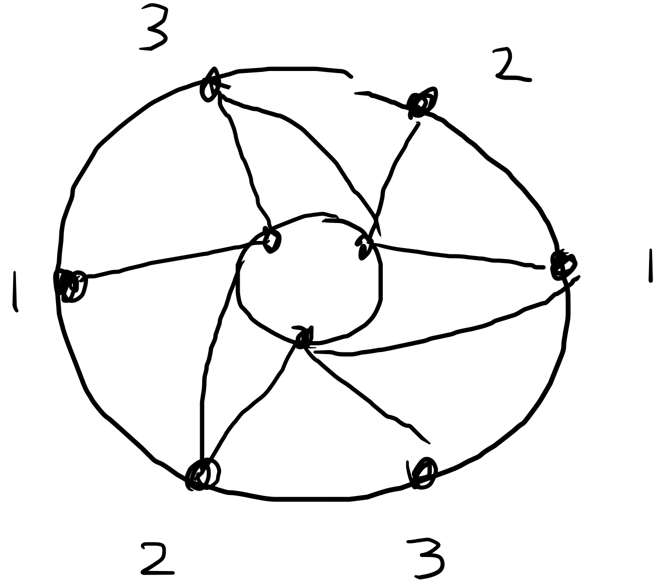
$$= (\eta_0 v_0 v_{i_1} \bar{\eta}_{i_1})(\eta_{i_1} v_{i_1} v_{i_2} \bar{\eta}_{i_2}) \cdots (\eta_{i_n} v_{i_n} v_0 \bar{\eta}_0)$$

$$\sim v_0 v_{i_1} \cdots v_{i_n} v_0$$

$\square$

## Example

K




$|K| \cong \mathbb{R}P^2$

**Ex** Compute $G(K;T)$

**Rmk** in general computing $G(K;T)$ is tedious — we will find a simpler way (van Kampen thm).

## II. Free groups & free products.

<u>Free groups</u> $S$ set (eg $S = \{a, b, c\}$)

free group $F(S) = \langle S \mid \rangle$ elements are reduced words in $S \cup S^{-1} \cup \{e\}$

eg $ac^{-1}ba^2c^3b$, $ae = a$, $eeee = e$, $ab^7b^{-7}a^{-1} = e$

operation: concatenate & reduce $(ac^2)(c^{-1}b) = ac^2c^{-1}b$
$= acb$.

$ab \neq ba$

# Free products

given groups $G, H$, the <u>free product</u> $G * H$

$G = \langle S \mid R \rangle$, $H = \langle S' \mid R' \rangle$

$G * H = \langle S, S' \mid R, R' \rangle$

<u>Ex</u>. $\mathbb{Z} * \mathbb{Z} = \langle a, b \mid \ \rangle \cong F(\{a, b\})$

$\mathbb{Z} = \langle a \mid \ \rangle$

$\mathbb{Z} = \langle b \mid \ \rangle$

**Ex** $G = H = \mathbb{Z}/2\mathbb{Z}$

$G = \langle a \mid a^2 = 1 \rangle$ $\quad$ $H = \langle b \mid b^2 = 1 \rangle$

$G * H = \langle a, b \mid a^2 = 1 = b^2 \rangle$

$b = b^{-1}, \; a = a^{-1}$

$abab \ldots, \quad baba \ldots$

- observation $\langle ab \rangle = \{ \ldots, baba, ba, e, ab, abab, \ldots \} \cong \mathbb{Z}$ is normal in $G$.

$a(ab)a^{-1} = ba$

$b(ab)b^{-1} = ba$

- $G * H / \langle ab \rangle = \; ?$

- There is a hom. $G * H \xrightarrow{\varphi} \underset{\{0,1\}}{\mathbb{Z}/2\mathbb{Z}}$

$$a \longmapsto 1$$
$$b \longmapsto 1.$$

$\ker(\varphi) =$ even length words. $= \langle ab \rangle$.

$\Longrightarrow \quad G * H / \langle ab \rangle \cong \mathbb{Z}/2\mathbb{Z}$

"short exact sequence"

$$a \longleftarrow\!| \ 1$$

$$1 \longrightarrow \underset{\langle ab \rangle}{\mathbb{Z}} \longrightarrow G * H \rightleftarrows \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

$\Longrightarrow \quad G * H = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \quad \left( \neq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \right)$

b/c ab & a don't commute.

Rmk $G * H$ is isomorphic to subgp of $\mathrm{Isom}(\mathbb{R})$ generated by

$A : x \longmapsto -x$ $\quad B : x \longmapsto -x+1$

$BA : x \longmapsto x+1$ translation

This group is the $\infty$ dihedral group.