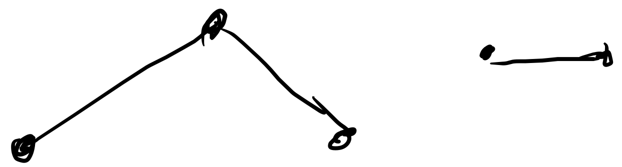
I. Computing edge groups

<u>Recap</u> K simplicial complex $K \subset P(V)$

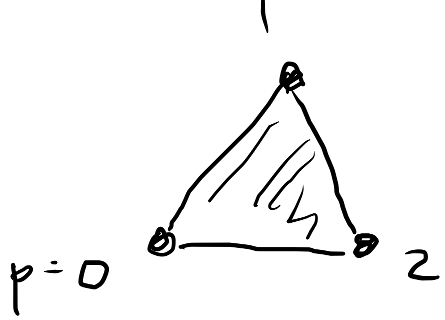
eg graphs: V vertex set $G \subset P(V)$ subsets of size ≤ 2.



given K, p vertex $\rightsquigarrow$ edge group

$$E(K,p) = \{\text{edge loops at } p\}/\sim$$

<u>Eg</u> $V = \{0,1,2\}$ $K = P(V) \setminus \emptyset$ $|K| =$



$E(K,p) \ni [0120], [0210], [01010]$.

these are all equivalent to $[0]$ eg $0120 \sim 020 \sim 0$.

Give a <u>presentation for $E(K,p)$.</u>

Assume K connected (any $u,v \in V$ are connected by an edge path)

Write $V = \{v_i\}$

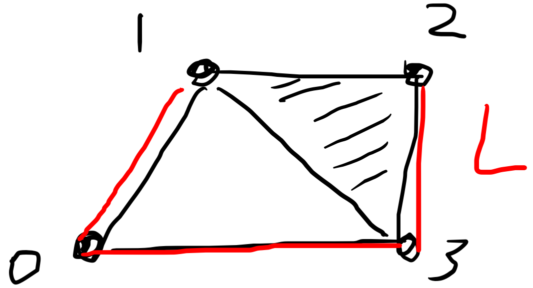
Choose max tree $L \subset K$ (cf. HW1 # 1)

<u>Define</u> $G(K,L)$ group with

- generators g_{ij} for each edge $\{v_i, v_j\}$ $i<j$
- relations
  - $g_{ij} g_{jk} = g_{ik}$ if $\{v_i, v_j, v_k\} \in K$ $i<j<k$
  - $g_{ij} = 1$ if $\{v_i, v_j\} \in L$.

Thm $G(K,L) \cong E(K,p)$

Ex



generators: $g_{01}, g_{03}, g_{12}, g_{13}, g_{23}$

relations: $g_{12} g_{23} = g_{13}$

$G(K,L) =$ $\quad g_{01} = 1, \quad g_{03} = 1, \quad g_{23} = 1$

$$\langle \cancel{g_{01}}, \cancel{g_{03}}, g_{12}, g_{13}, \cancel{g_{23}} \mid g_{01} = g_{03} = g_{23} = 1, \; g_{12} \cancel{g_{23}} = g_{13} \rangle$$

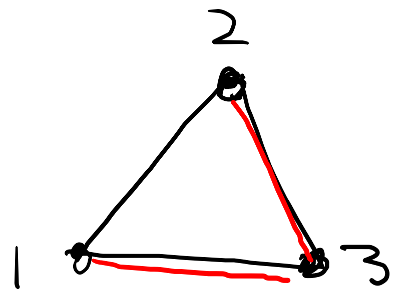
$$\cong \langle g_{12} \mid \; \rangle \cong \mathbb{Z}.$$

Rmk Thm gives reasonable way to compute $\pi_1(X)$ if X has a triangulation $X \cong |K|$

since $\pi_1(X) \cong \pi_1(|K|) \cong E(K,p) \cong G(K,L)$

need proof

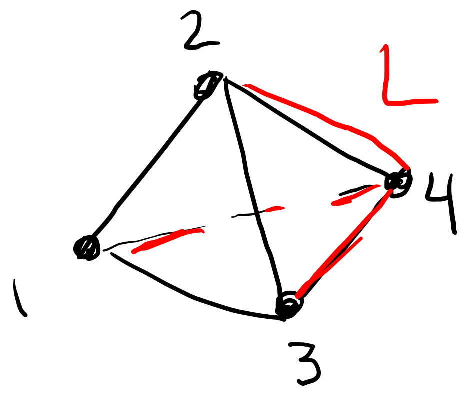
Ex $|K| \cong S^1$



$$G(K,L) = \langle g_{12}, g_{13}, g_{23} \mid g_{13} = 1 = g_{23} \rangle$$

$$= \langle g_{12} \mid \rangle \cong \mathbb{Z}$$

__Ex__ $|K| \cong S^2$



$$G(K,L) = \langle \ | \ \rangle = \{1\}.$$

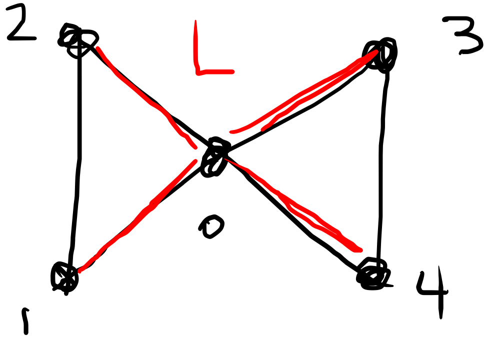
generators $\cancel{g_{12}}$ $\cancel{g_{13}}$ $\cancel{g_{14}}$ $\cancel{g_{23}}$ $\cancel{g_{24}}$ $\cancel{g_{34}}$

relations $g_{12} g_{23} = g_{13}$ $\quad g_{12} \cancel{g_{24}} = \cancel{g_{14}}$

$g_{13} \cancel{g_{34}} = \cancel{g_{14}}$ $\quad g_{23} \cancel{g_{34}} = \cancel{g_{24}}$

$g_{14} = 1$ $\quad g_{24} = 1$ $\quad g_{34} = 1$

Ex $|K| \cong S^1 \vee S^1$



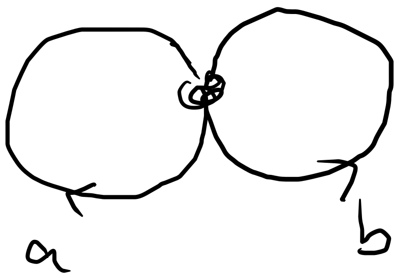
$G(K,L) = \langle g_{12}, g_{34} \mid \rangle$

generators $g_{12}, \cancel{g_{01}}, \cancel{g_{02}}, \cancel{g_{03}}, \cancel{g_{04}}, g_{34}$

relations $g_{01} = 1 = g_{02} = g_{03} = g_{04}$

a group with no relations is called a <u>free group</u>.

$\langle a, b \mid \rangle \ni a b a^{-1} b^{2} a^{-7} b$



Rmks

- $G(K,L)$ doesn't use any info about simplices of K of dim ≥ 3 (only use vertices, edges, faces)

- it can be tedious to write down $G(K,L)$ if K has many simplices. The presentation for $G(K,L)$ is often very redundant.

- This is problematic! $\nexists$ algorithm to decide if two presentations $\langle S \mid R \rangle$ $\langle S' \mid R' \rangle$ define isomorphic groups. eg can't decide in general if $\langle S \mid R \rangle = \{1\}$.

- For different choices of $L \subset K$, get different presentations: $G(K,L) \overset{?}{\cong} G(K,L')$

<u>Thm</u> $E(K,p) \cong G(K,L)$