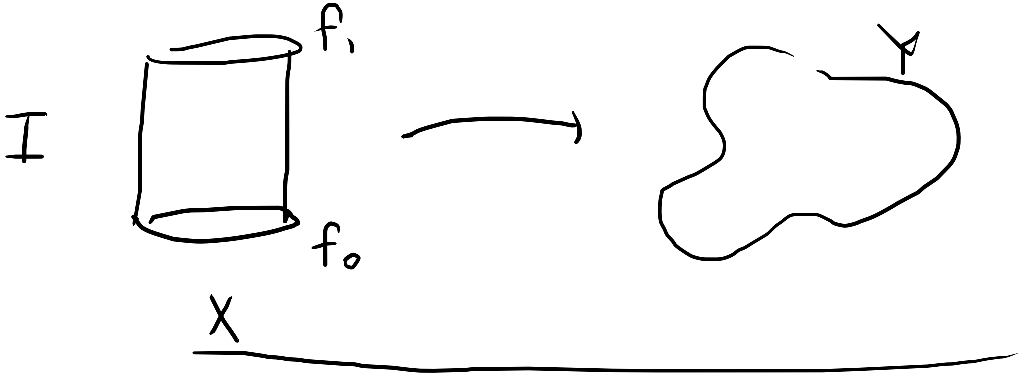
I. Homotopy equivalences and π_1

Previously $X \cong Y \implies \pi_1(X) \cong \pi_1(Y)$

__Defn__ maps $f_0, f_1 : X \longrightarrow Y$ are __homotopic__ $f_0 \sim f_1$ if

$\exists \; h : X \times I \longrightarrow Y$ s.t. $h|_{X \times \{i\}} = f_i$ for $i = 0, 1$.



Say X, Y are __homotopy equivalent__ $X \simeq Y$

if $\exists$ $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t.

$g \circ f \sim id_X$ and $f \circ g \sim id_Y$

Examples

(1) $X \cong Y \Rightarrow X \simeq Y$

(2) $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$

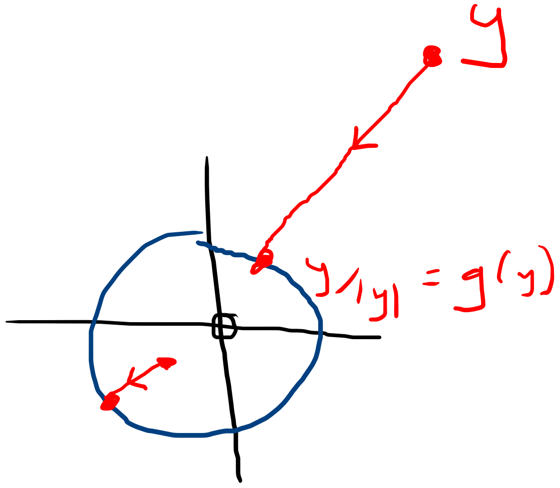
$$S^1 \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{R}^2 \setminus \{0\}$$

$f(x) = x$

$g(y) = \frac{y}{|y|}$

$g \circ f(x) = g(x) = \frac{x}{|x|} = x \Rightarrow g \circ f = id_{S^1}$

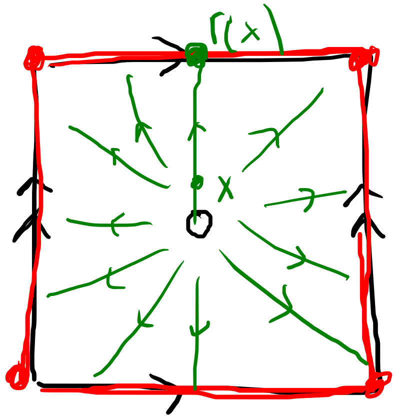
$f \circ g(y) = f\left(\frac{y}{|y|}\right) = \frac{y}{|y|}$

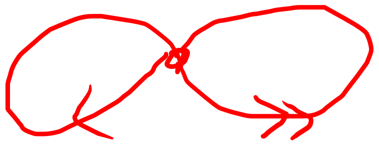


$f \circ g \sim id_{\mathbb{R}^2 \setminus 0}$ via straight line homotopy

$h(y,t) = (1-t)y + t\frac{y}{|y|}$.

- $T^2 \setminus pt \simeq S^1 \vee S^1$





$$\begin{tikzcd} S^1 \vee S^1 \arrow[r, bend left, "\text{inclusion}"] & T^2 \setminus pt \arrow[l, bend left, "r"] \end{tikzcd}$$

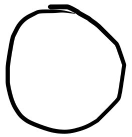
"project radially"

Thm $X \simeq Y \implies \pi_1(X) \cong \pi_1(Y)$

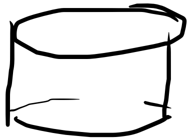
"π_1 is a homotopy invariant"

Rmk This is reasonable since π_1 = homotopy classes of loops

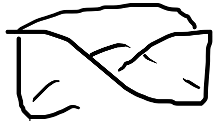
Ex These spaces are h.e. ($\simeq$) (HW 7)

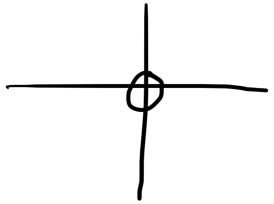
$S^1 =$ 

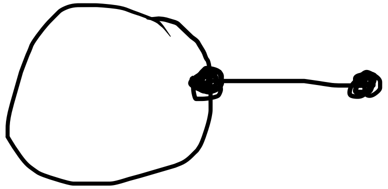
hence have

annulus 

$\pi_1 = \mathbb{Z}$

Möbius 

$\mathbb{R}^2 \setminus \{0\}$ 

$C =$  $= S^1 \vee I$

II. Fundamental Theorem of Algebra

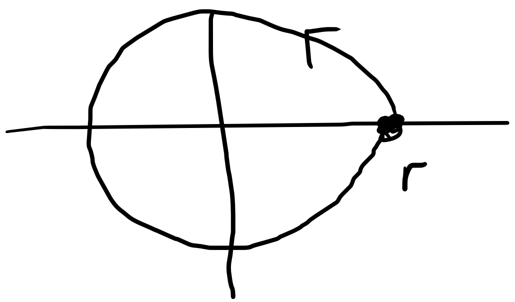
<u>Thm</u> (FTA) Fix $n \geq 1$. A polynomial

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \qquad a_i \in \mathbb{C}$$

has a root in $\mathbb{C}$, ie $\exists \; \beta \in \mathbb{C}$ st. $p(\beta) = 0$.

<u>Topology proof</u> (sketch). By contradiction suppose p has no root.

- For $r > 0$ let $C_r : [0,1] \to \mathbb{C} \setminus \{0\}$ $\quad C_r(t) = r e^{2\pi i t}$



If p has no root then

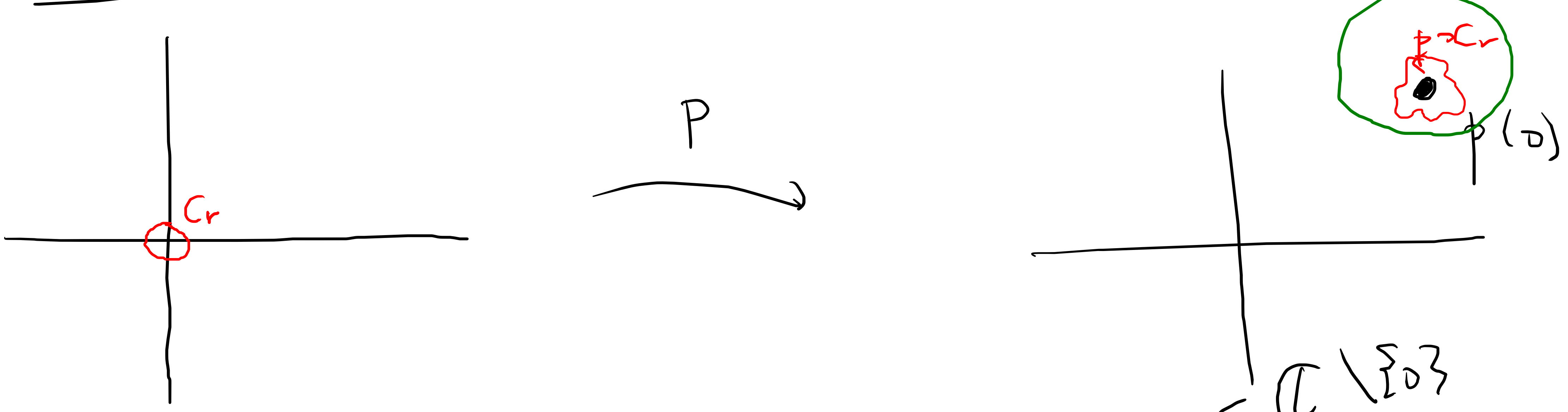
$p \circ C_r$ is a loop in $\mathbb{C} \setminus \{0\}$.

$\Rightarrow p \circ C_r$ defines an element of $\pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$ for each r.

$p \circ C_r \sim p \circ C_{r'}$ homotopic loops,

so for every r we get <u>same</u> element of $\pi_1(\mathbb{C} \setminus \{0\})$.

Claim 1 For $r << 0$ $C_r \sim$ constant.



By continuity $p \circ C_r \subset B_\varepsilon(p(0)) \subset \mathbb{C} \setminus \{0\}$. Can homotope C_r to constant in this ball.

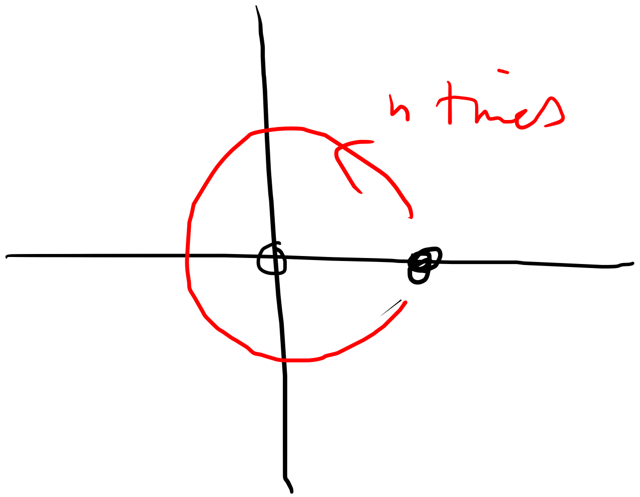
Claim 2 For $r >> 1$

$p \circ C_r$ homotopic to

$f : t \mapsto (re^{2\pi i t})^n$

exercise: (straight line homotopy)

(homotopy of maps $[0,1] \longrightarrow \mathbb{C} \setminus \{0\}$.)



$f \sim e^{2\pi i t \cdot n}$

Claims 1 + 2 $\Longrightarrow$

loop $t \mapsto e^{2\pi i n t}$ homotopic to a constant

$\times$.

$\square$