I. Path lifting and $\pi_1(S^1)$

<u>Last time</u> $\pi_1(S^n) = 0$ for $n \geq 2$.

<u>Thm</u> $\pi_1(S^1, 1) \cong \mathbb{Z}$.

<u>Path lifting</u>. Consider
$$p : \mathbb{R} \longrightarrow S^1 \subset \mathbb{C}$$
$$t \longmapsto e^{2\pi i t}$$

<u>Prop</u> (path lifting) For any loop $f : [0,1] \longrightarrow S^1$ based at 1. there exists unique $\tilde{f} : [0,1] \longrightarrow \mathbb{R}$ st. $\tilde{f}(0) = 0$ and $p \circ \tilde{f} = f$

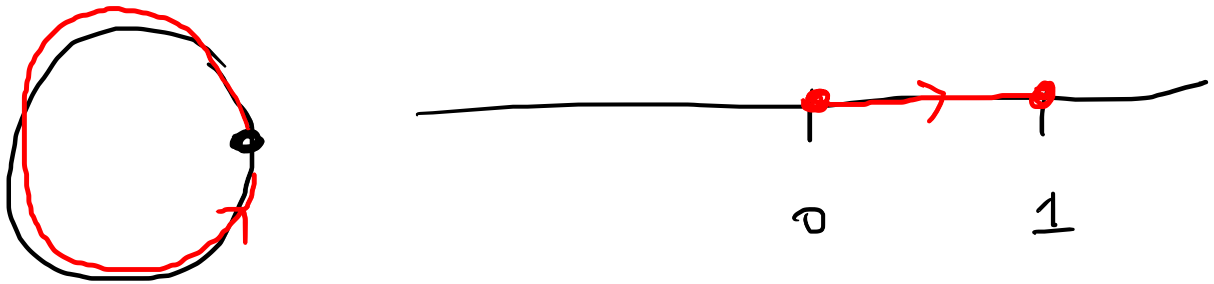
we call $\tilde{f}$ a <u>lift</u> of f if $p \circ \tilde{f} = f$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Examples

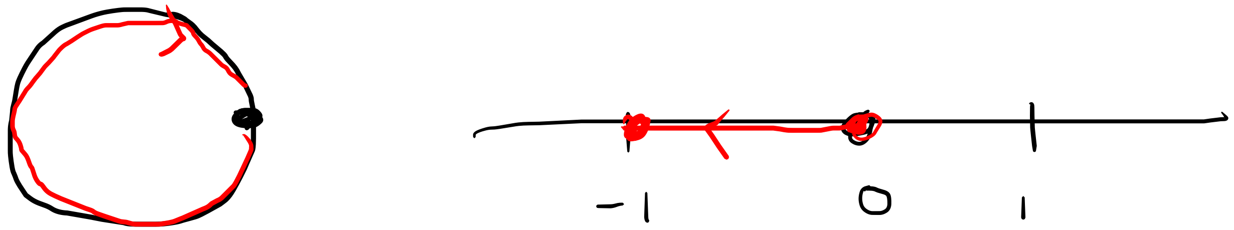
- $f(t) = e^{2\pi i t}$,

  $\tilde{f}(t) = t$

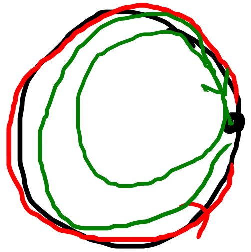


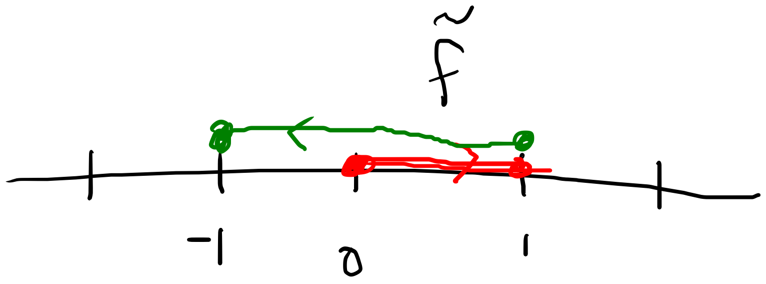
- $f(t) = e^{-2\pi i t}$

  $\tilde{f}(t) = -t$



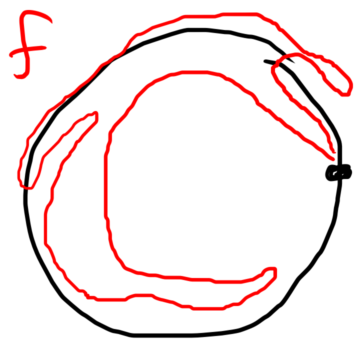
- $$f(t) = \begin{cases} e^{4\pi i t} \ f & 0 \le t \le 1/2 \\ e^{-8\pi i t} \ g & 1/2 \le t \le 1. \end{cases}$$

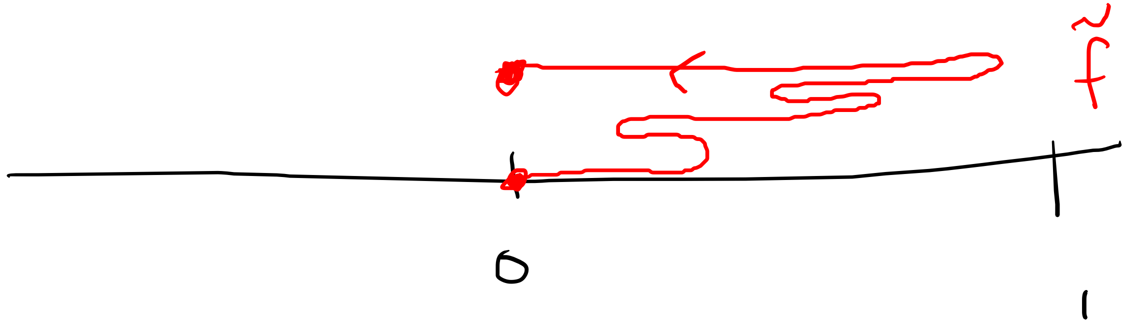




$$\tilde{f}(t) = \begin{cases} 2t & t \in [0, 1/2] \\ 3 - 4t & t \in [1/2, 1] \end{cases}$$

$e^{-4\pi i}$ $e^{-8\pi i}$

• 



Rmks • If $\tilde f$ lift of f with $\tilde f(0) = 0$.

then $\tilde{\tilde f}(t) = \tilde f(t) + n$ is also a lift of f

↳ p is periodic

$e^{2\pi i(t+k)} = e^{2\pi i t}$

$k \in \mathbb{Z}$.

but with $\tilde{\tilde f}(0) = n$.

• What can we say about $\tilde f(1)$

$$p(\tilde f(1)) = f(\underset{\substack{\cap \\ [0,1]}}{1}) = \underset{\substack{\cap \\ S^1}}{1} \Rightarrow \tilde f(1) \in p^{-1}(1) = \mathbb{Z}.$$

$f \longmapsto \tilde{f}(1)$ is like the "winding number" of f
$\tilde{f}(1) \in \mathbb{Z}$.

<u>WTS</u> $f \longmapsto \tilde{f}(1)$ defines a isomorphism $\pi_1(S^1, 1) \longrightarrow \mathbb{Z}$.

- well-defined on homotopy classes

ie if $f \sim g$, then $\tilde{f}(1) = \tilde{g}(1)$

idea: given a homotopy f_t of loops in S^1 at 1
can lift to get homotopy $\tilde{f}_t$ of paths in $\mathbb{R}$ w/ $p \circ \tilde{f}_t = f_t$
then $\tilde{f}_t(1) \in \mathbb{Z}$. By continuity $\tilde{f}_t(1)$ constant ($\mathbb{Z}$ b/c discrete)
so $\tilde{f}_1(1) = \tilde{f}_0(1)$

Thm $\pi_1(S^1, 1) \xrightarrow{\Phi} \mathbb{Z}$ is a group isomorphism

$[f] \longmapsto \tilde{f}(1)$

Proof

① homomorphism WTS: $\widetilde{f * g}(1) = \tilde{f}(1) + \tilde{g}(1)$.

observe $\widetilde{f * g} = \tilde{f} * \tilde{\tilde{g}}$ where $\tilde{\tilde{g}}(t) = \tilde{g}(t) + \tilde{f}(1)$.

Then $\widetilde{f * g}(1) = \tilde{\tilde{g}}(1) = \tilde{g}(1) + \tilde{f}(1)$.

② surjective $f_k(t) = e^{k(2\pi i t)}$

has $\tilde{f}_k(t) = kt$ so $\tilde{f}_k(1) = k$.

③ Suppose $\tilde{f}(1) = 0$ (WTS: $f \sim$ constant)

Note $\tilde{f}$ is a loop in $\mathbb{R}$ based at 0.

Know $\pi_1(\mathbb{R}) = 0$. Choose homotopy $(\tilde{f})_t$ from $\tilde{f}$ to constant $0 \in \mathbb{R}$

Now define

$f_t = p \circ (\tilde{f})_t$. This is a homotopy between

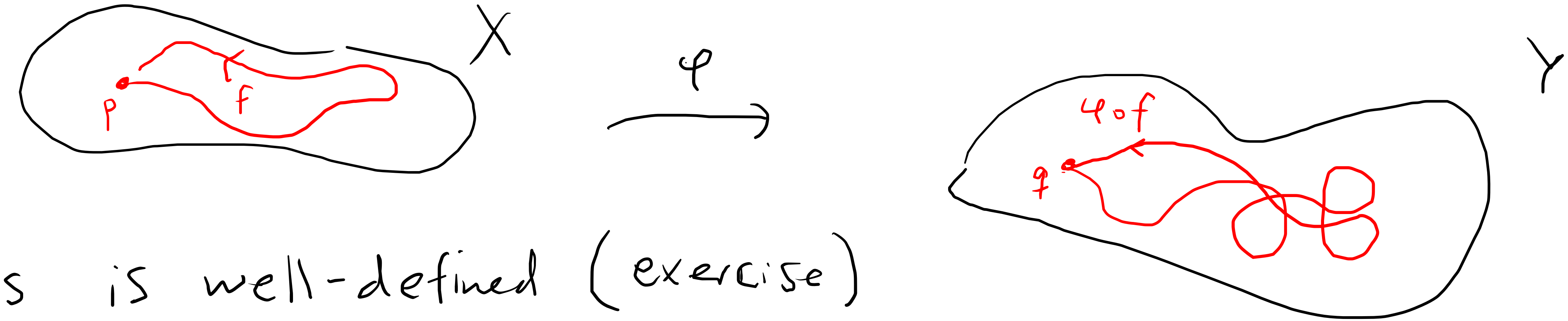
$p \circ (\tilde{f})_0 = p \circ \tilde{f} = f$ and $p \circ (\tilde{f})_1 = 1$.

$\square$

II. Induced maps

$\varphi : X \longrightarrow Y \qquad p \in X, \qquad q = \varphi(p) \in Y.$

Define the <u>induced map</u>
$$\varphi_* : \pi_1(X, p) \longrightarrow \pi_1(Y, q)$$
$$[f] \longmapsto [\varphi \circ f]$$



This is well-defined (exercise)

and φ_* is a homomorphism:

$$\varphi_*([f]\cdot[g]) = \varphi_*([f*g])$$

$$= [\varphi\circ(f*g)]$$

$$= [(\varphi\circ f)*(\varphi\circ g)]$$

$$= [\varphi\circ f]\cdot[\varphi\circ g]$$

$$= \varphi_*([f])\cdot\varphi_*([g])$$

$$\varphi(f*g(t)) = \begin{cases} \varphi(f(2t)) & \cdots \\ \varphi(g(2t-1)) & \cdots \end{cases}$$

$$\parallel$$

$$(\varphi\circ f)*(\varphi\circ g)\ (t)$$

$\checkmark$

Ex $\varphi : S^1 \longrightarrow S^1 \qquad \varphi(e^{i\theta}) = e^{2i\theta}$

$\pi_1(S^1, 1) \cong \mathbb{Z}$ generated by $f(t) = e^{2\pi i t}$

To determine $\varphi_* : \mathbb{Z} \longrightarrow \mathbb{Z}$ compute
$1 \longmapsto 2$

$\varphi_*([f]) = [\varphi \circ f]$

$$\varphi(f(t)) = \varphi(e^{2\pi i t}) = e^{4\pi i t} \longleftrightarrow 2 \in \mathbb{Z}$$
$$= f * f$$

So φ_* is mult by 2.

Properties of induced maps

(1) $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$

then

$$\begin{tikzcd}
{[f] \in \pi_1(X)} \arrow[r, "\varphi_*"] \arrow[dr, "(\psi \circ \varphi)_*"'] & \pi_1(Y) \arrow[d, "\psi_*"] \\
 & \pi_1(Z)
\end{tikzcd}$$

Claim $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$

$$(\psi \circ \varphi)_*([f]) = [(\psi \circ \varphi) \circ f] = [\psi \circ (\varphi \circ f)] = \psi_*(\varphi_*([f]))$$

✓

(2) if $\varphi = id_X : X \to X$ (Functoriality)

then $\varphi_* = id_{\pi_1(X)}$

<u>Cor.</u> $\pi_1(-)$ is a topological invariant.

ie $X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$

<u>Proof</u> $X \cong Y \Rightarrow \exists$

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & Y \arrow[l, bend left, "\psi"] \end{tikzcd}$$

$\psi \circ \varphi = id_X$

$\varphi \circ \psi = id_Y$.

$\Rightarrow \psi_* \circ \varphi_* = (\psi \circ \varphi)_* = (id_X)_* = id_{\pi_1(X)}$

and $\varphi_* \circ \psi_* = id_{\pi_1(Y)} \Longrightarrow \varphi_* : \pi_1(X) \to \pi_1(Y)$ is iso. $\square$