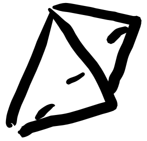
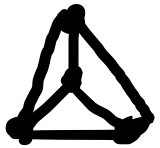
I. Proof of Euler's Thm

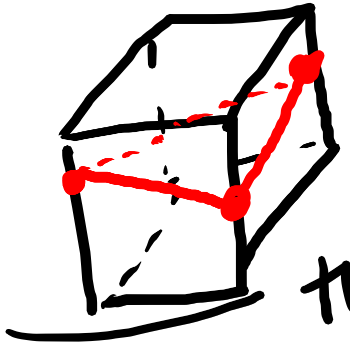
Recap

- Polyhedron $P \rightsquigarrow$ graph G
  ↳ 1-skeleton

$P =$ 

$G =$ 

- Euler's Thm: P polyhedron with 1-skeleton G.
  If (i) G connected and (ii) any polygon loop (not necessarily edges) separates P into two pieces
  then $V_P - E_P + F_P = 2$.



- Every graph G has a tree $T \subset G$ that contains every vertex HW1
- For a tree $V_T - E_T = 1$.

Euler's Thm: P polyhedron with 1-skeleton G

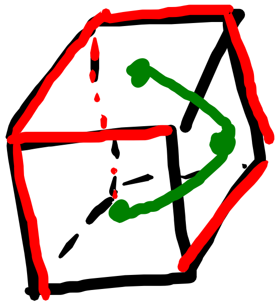
If (i) G connected and (ii) any polygonal loop (not necessarily edges) separates P into two pieces

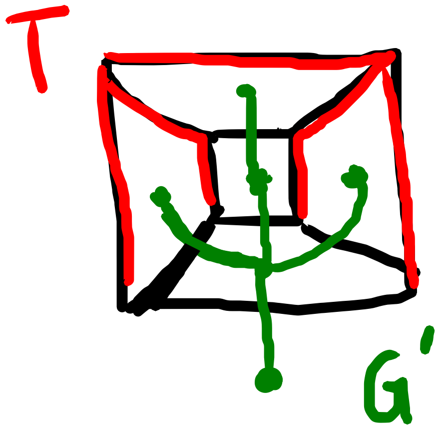
then $V_P - E_P + F_P = 2$.

Proof **Step 1** (dual graph) Choose $T \subset G$ contains every vertex

Define "dual graph" G' : vertices $\longleftrightarrow$ faces of P

edges $\longleftrightarrow$ faces meet along edge not in T.

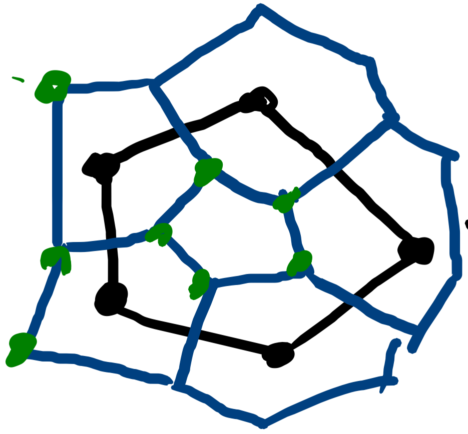
$P =$ 



<u>Claim</u> G' is a tree (!)

<u>Proof of Claim</u>: Suppose for contradiction that G' not a tree. Then $\exists$ nonbacktracking loop γ in $G' \subset P$. By assumption γ separates P.

↳ This contradicts the fact that T is connected, contains every vertex of P and is disjoint from $G' \supset \gamma$



$\gamma \subset G'$

Therefore G' is a tree.

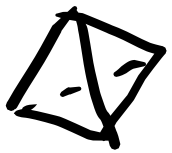
Step 2 Compute

$$V_P - E_P + F_P = V_T - (E_T + E_{G'}) + V_{G'}$$

$$= \underbrace{(V_T - E_T)}_{1} + \underbrace{(V_{G'} - E_{G'})}_{1} = 2.$$

□

Rmk For all the (limited number of) examples we've seen ~~with $\chi(P) = 2$~~ satisfying assumptions of Euler's thm P is "topologically S^2".

 $\leftrightarrow S^2$

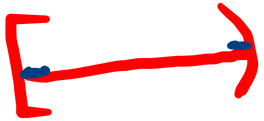
Is this always the case?

II. Topological equivalence

<u>Defn</u> $X, Y \subset \mathbb{R}^n$ are <u>topologically equivalent</u> if $\exists$ <u>continuous</u> bijection $f: X \to Y$ with continuous inverse $f^{-1}: Y \to X$. Write $X \cong Y$.

- intuitively f continuous means if $x_1, x_2 \in X$ "close" then $f(x_1), f(x_2) \in Y$ "close"

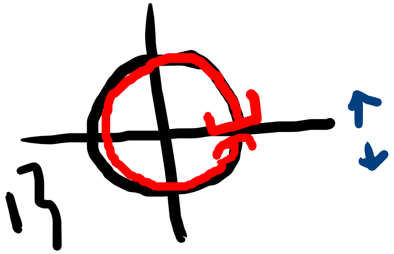
<u>Nonexample</u> $X = [0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$



$f: X \to Y$

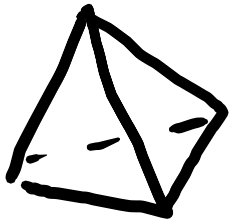
$x \mapsto (\cos(2\pi x), \sin(2\pi x))$

$Y = S^1$
$= \{(x,y) \mid x^2 + y^2 = 1\}$



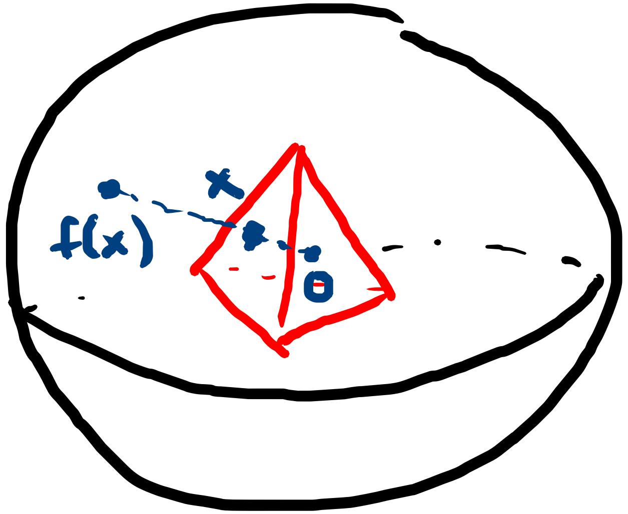
f^{-1} not continuous

Example

 $\cong S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$.

Pf: radial projection

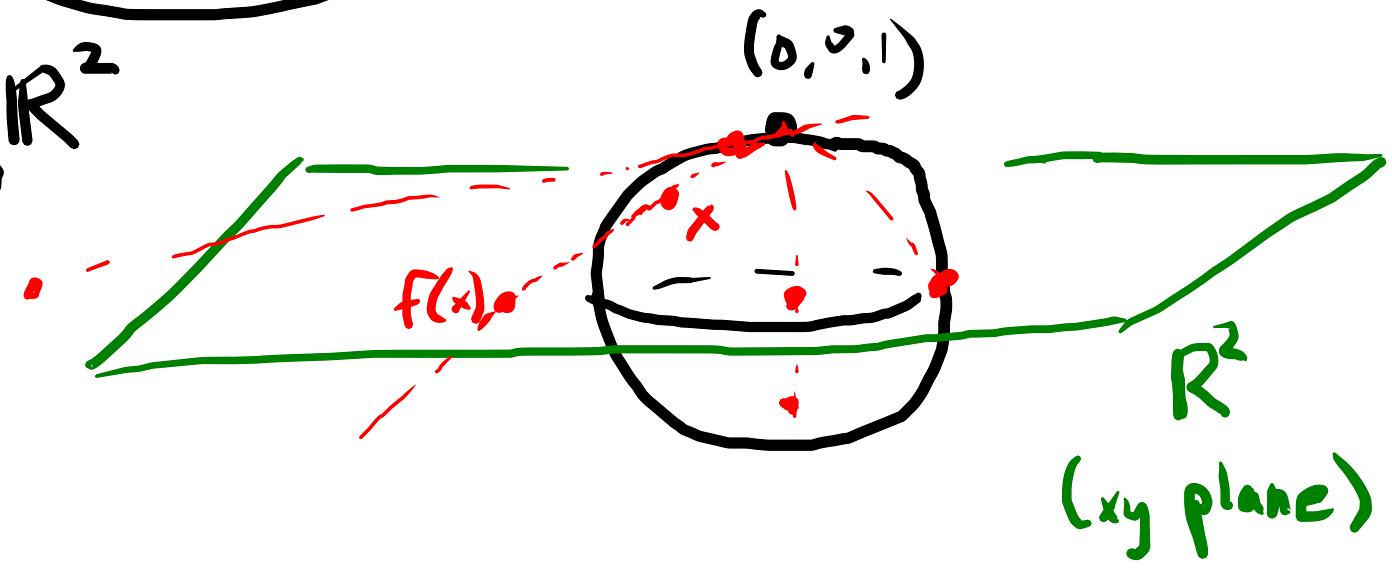
$$f(x) = \frac{x}{|x|}$$



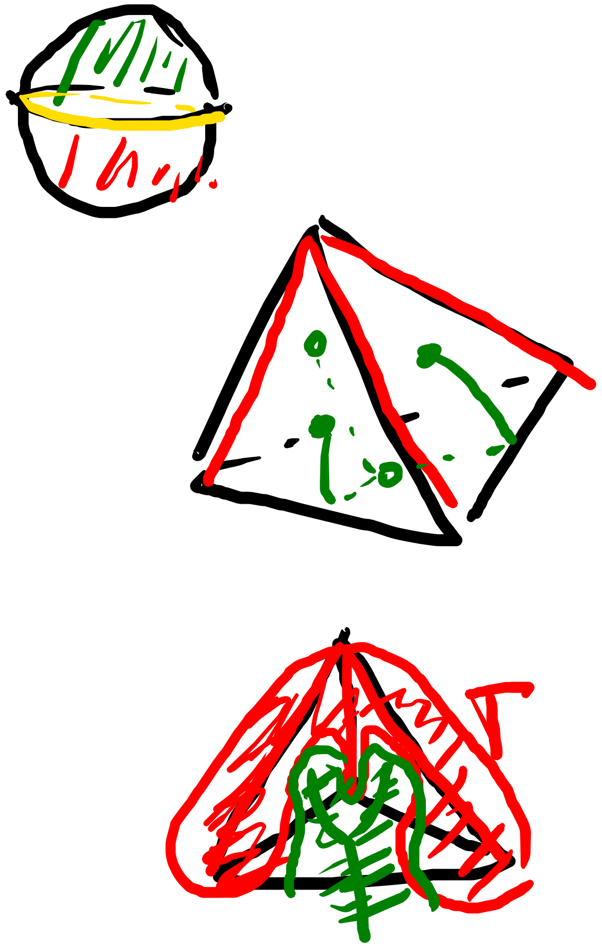
Example

$S^2 \setminus \{(0,0,1)\} \underset{f}{\cong} \mathbb{R}^2$

Pf stereographic projection



Thm P polyhedron satisfying assumptions of Euler's thm. (1-skel. connected, every polygonal loop separates). Then $P \cong S^2$.



Proof idea use proof of Euler's thm.

Found complementary trees $T, G' \subset P$

Thick these to disks until they meet.

$\Rightarrow P =$ union of two disks glued along boundary circle

$= S^2$