I. Fundamental groups of spheres.

Defn. A path connected space X is called **Simply connected** if $\pi_1(X, p) = \{1\}$ $\forall\, p \in X$.

Today Examples.

Lemma. $\pi_1(\mathbb{R}^n, 0) = \{1\}$

Proof. WTS every loop $f: [0,1] \to \mathbb{R}^n$ based at 0 is nullhomotopic

$$f_t(s) = (1-t) f(s). + t \cdot 0$$

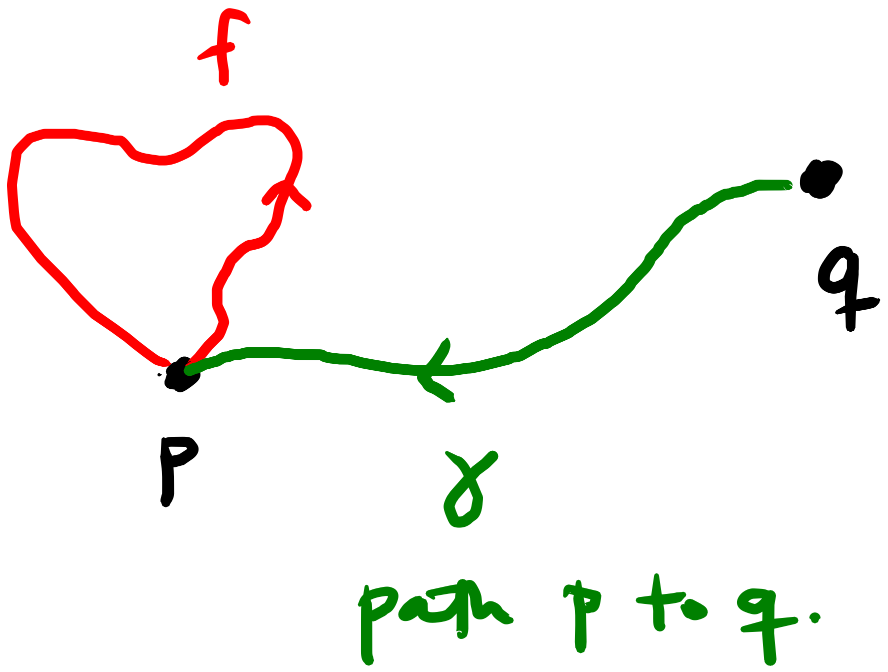
$\square$

Same argument work for any other basepoint.

Lemma (later) X path connected

$\Rightarrow \pi_1(X,p) \cong \pi_1(X,q) \quad \forall\, p,q \in X.$

Rmk Consequently often write $\pi_1(X)$ instead of $\pi_1(X,p)$



$$\pi_1(X,p) \xrightarrow{\Phi} \pi_1(X,q)$$

$$[f] \longmapsto [\gamma * f * \bar{\gamma}]$$

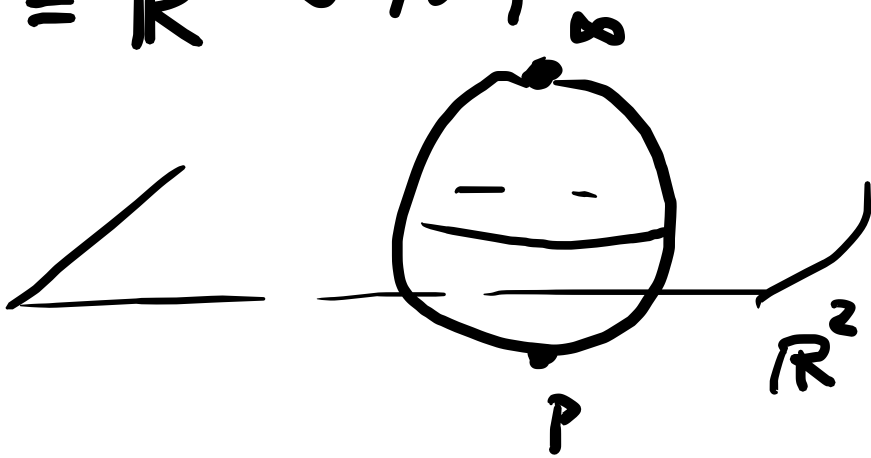
(later show Φ is $\cong$)

Thus to show X is simply connected, suffices to prove $\pi_1(X, p) = \{1\}$ for some $p \in X$.

Thm Fix $n \geq 2$. Then $\pi_1(S^n) = \{1\}$

For concreteness focus on S^2.

Idea: via stereographic projection $S^2 = \mathbb{R}^2 \cup \{\infty\}$



WTS every $f : [0,1] \to S^2$ based at p is nullhomotopic.

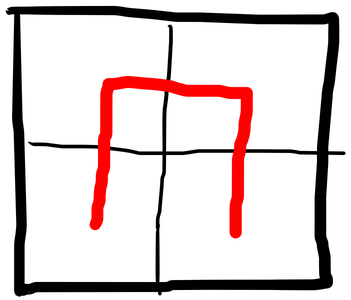
Observation: if $\infty \notin f([0,1])$ then $f([0,1]) \subset \mathbb{R}^2$ and can use fact that loops in $\mathbb{R}^2$ are nullhomotopic.

Problem ∃ continuous surjections $[0,1] \to S^2$!

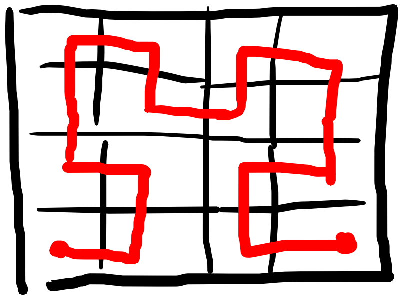
"Space filling curve"

Claim ∃ $\overset{\text{cts}}{\text{surjection}}$ $f : [0,1] \longrightarrow [0,1]^2$.

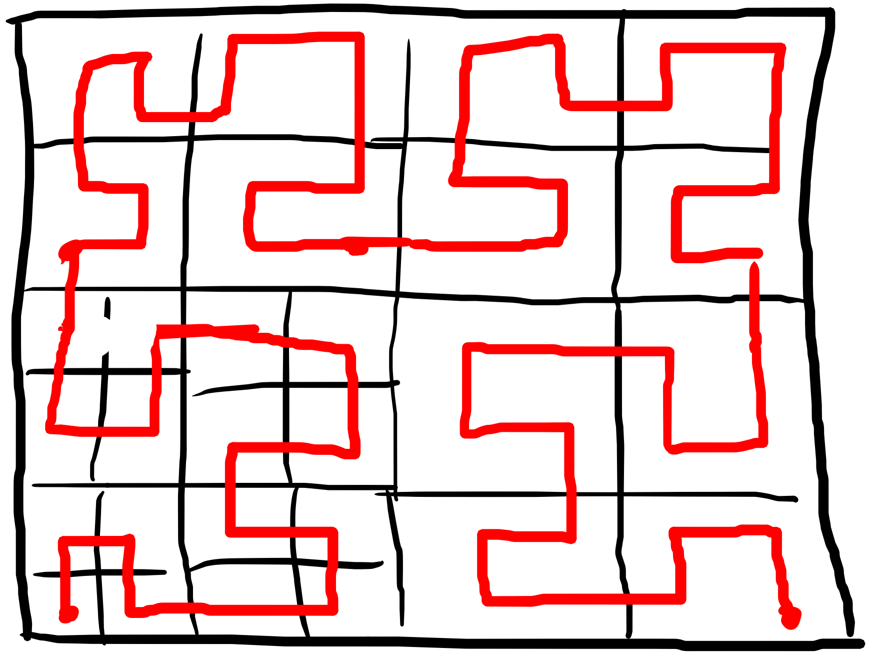
idea: define f as a "limit of sequence of $f_n : [0,1] \to [0,1]^2$"



f_1



f_2



f_3

. . .

By "analysis" (Arzela-Ascoli)

f_n converge in $C([0,1],[0,1]^2)$

(Space of continuous maps, topologized w/ metric

$$d(g_1,g_2) = \sup_{t\in[0,1]} d_{[0,1]^2}(g_1(t),g_2(t)))$$

to continuous $f:[0,1] \longrightarrow [0,1]^2$.

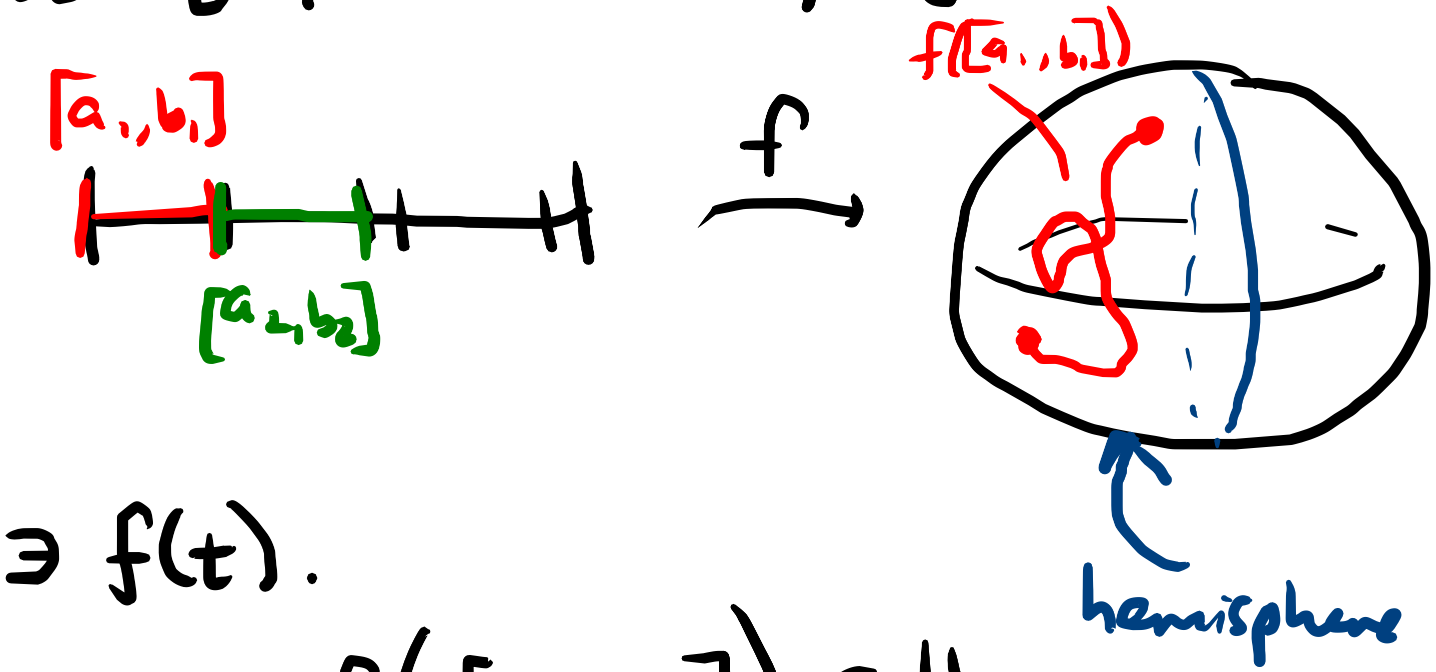
<u>Claim</u> f is surjective: image of f is dense in $[0,1]^2$ and image is compact hence closed.

$\Rightarrow$ image $= [0,1]^2$ $\square$

Proof of Thm $\pi_1(S^2) = \{1\}$.

Strategy Show any loop $f:[0,1] \longrightarrow S^2$ can be homotoped to a map that's not surjective.

Step 1 We can decompose $[0,1] = \bigcup [a_i,b_i]$ so that $f([a_i,b_i]) \subset$ some hemisphere



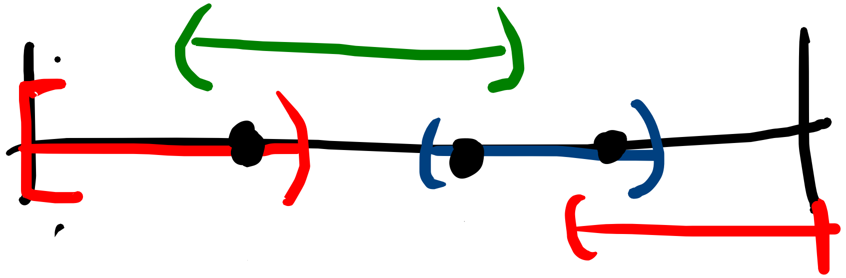
Indeed: $\forall t \in [0,1]$

choose open hemisphere $H_t \ni f(t)$.

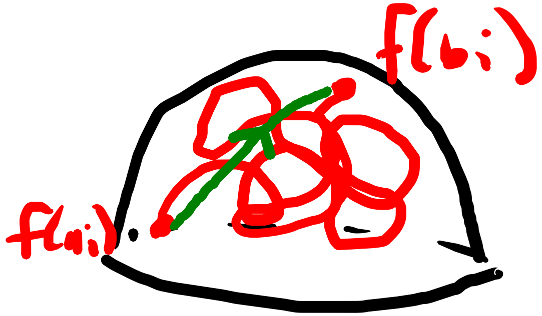
By continuity $\exists\ [a_t,b_t] \ni t$ s.t. $f([a_t,b_t]) \subset H_t$

Then $[0,1] = \bigcup_{t\in[0,1]} \left(\frac{a_t}{2}, \frac{b_t}{2}\right)$ is open cover

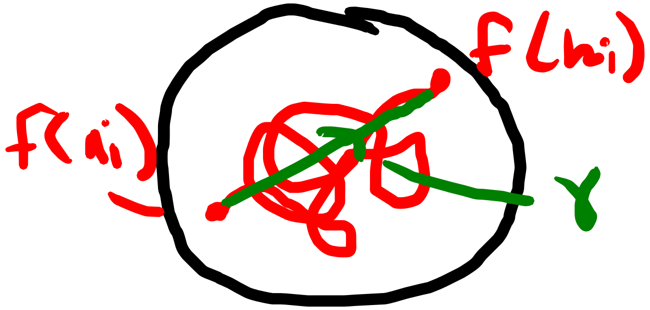
$[0,1]$ compact $\Rightarrow$ $\exists$ finite subcover



Step 2 Homotope $f|_{[a_i,b_i]}$ so image is a great circle arc. (ie $S^2 \cap \mathbb{R}^2$ ← plane through origin in $\mathbb{R}^3$)



$H \cong \mathbb{D}^2$



$t\in[a_i,b_i]$

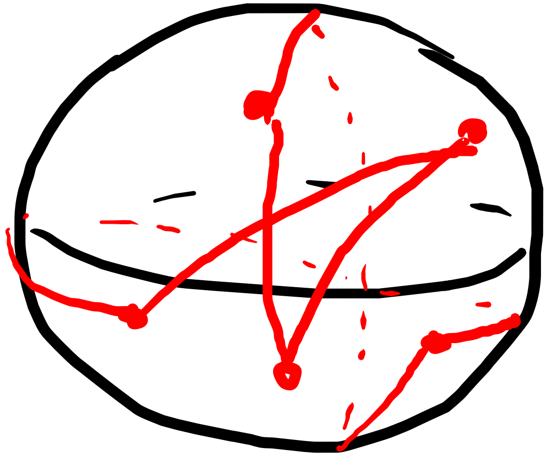
$$f_s(t) = (1-s)\, f(t) + s\cdot \gamma(t)$$

$$f_s(a_i) = (1-s)\, f(a_i) + s\cdot f(a_i) = f(a_i)$$

(Important: image of a_i, b_i don't change under the homotopy.)

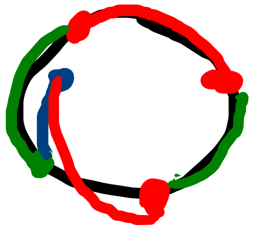
Step 3 After this homotopy $f([0,1])$ is union of finitely many great circle arcs

$\Rightarrow$ f not surjective (exercise)



$\Rightarrow$ can homote f to constant in $\mathbb{R}^2 \subset S^2$. $\square$

Rmk For $n=1$ i.e. S^1, argument shows any loop can be homotoped to a union of linear arcs.



here can't conclude f not surj.

$\pi_1(S^1) \cong \mathbb{Z}$.