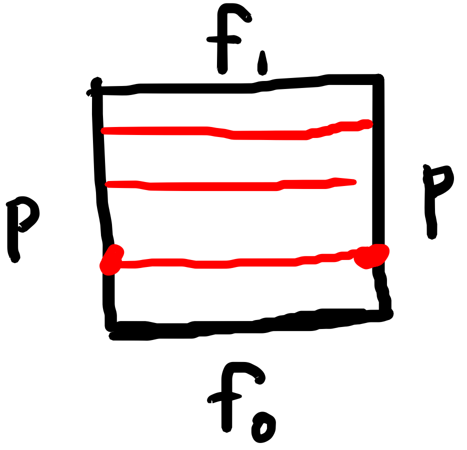
# I. Defining the fundamental group

**Recall** $(X, p)$ based space

$f : [0,1] \longrightarrow X$ loop $\quad f(0) = f(1) = p$

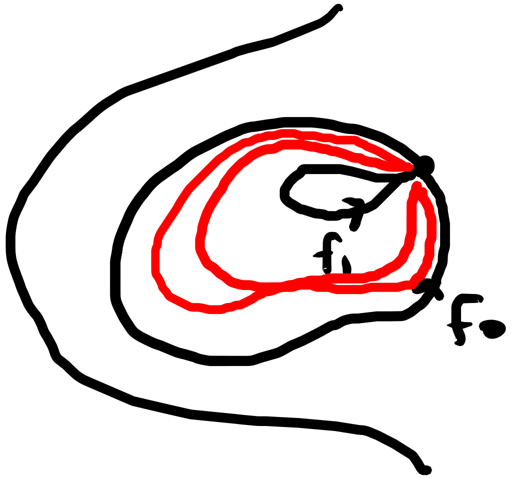
$F : [0,1]^2 \longrightarrow X$ **homotopy** between $f_0, f_1$ loops if

$F(s, i) = f_i$ for $i = 0,1$, $\quad F(0,t) = p = F(1,t)$




$f_t := f(-, t) : [0,1] \longrightarrow X$ loop

varying $t$ gives "path in the space of loops" from $f_0$ to $f_1$




write $f_0 \sim f_1$ homotopic.

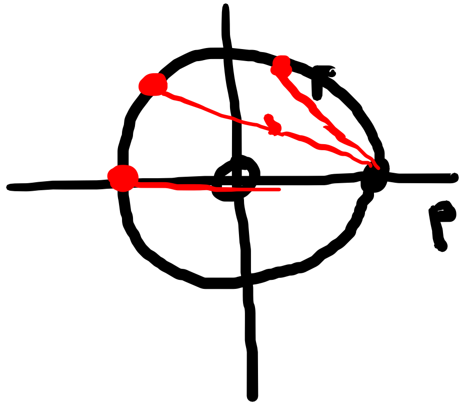
__Examples__ $X = \mathbb{R}^2$, $Y = \mathbb{R}^2 \setminus \{(0,0)\}$, $Z = \mathbb{R}^2 \setminus \{(0,0),(0,1)\}$

$p = (1,0)$

(a) Any loop $f : [0,1] \to X$ homotopic to a constant ("is nullhomotopic") via "straight-line homotopy"

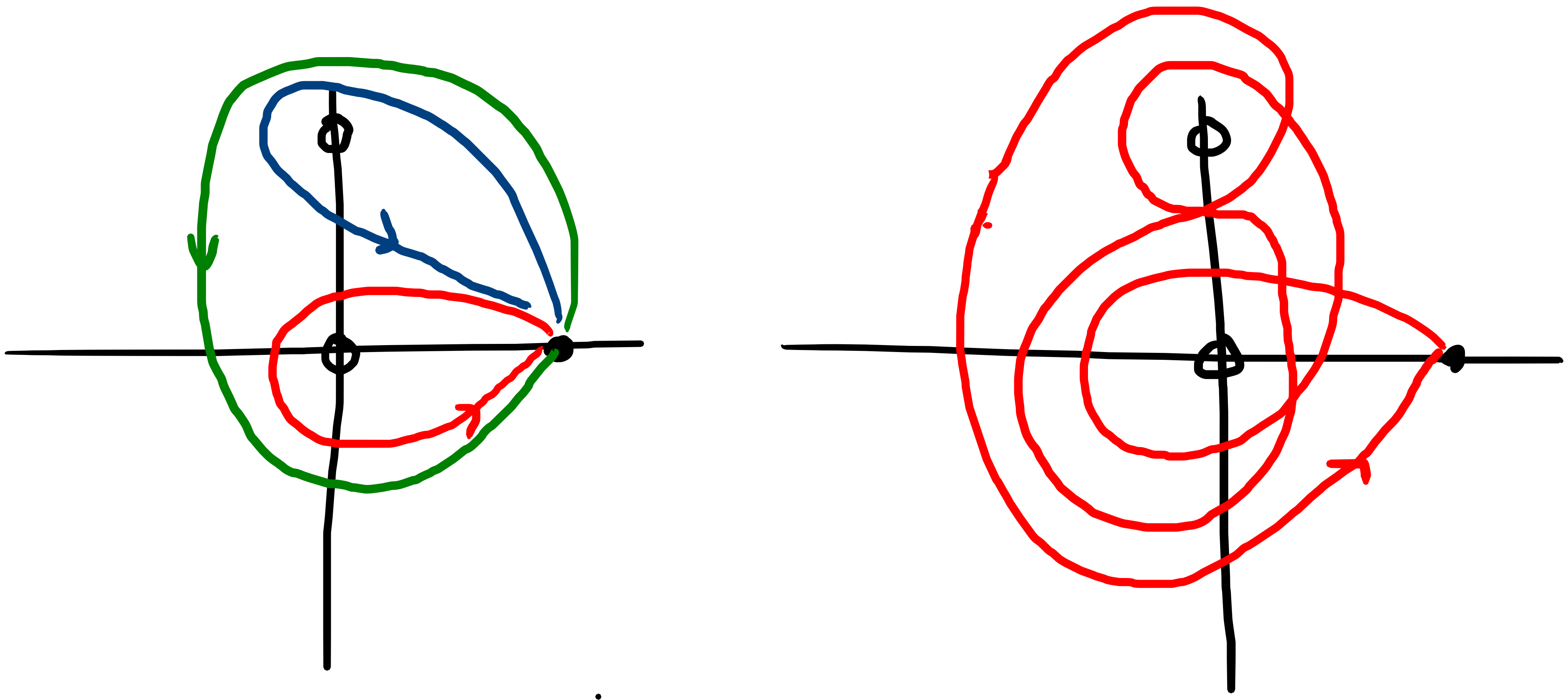
$$F(s,t) = (1-t)f(s) + tp$$

(b) $g : [0,1] \to Y$ $\quad g(s) = (\cos 2\pi s, \sin 2\pi s)$




$g$ not homotopic to constant in $Y$.

(c) $Z$ has many loops, distinct up to homotopy



Defined $\pi_1(X, p) = \{ \text{loops } f : [0,1] \to X \text{ based at } p \} / \sim$

Denote $[f]$ equivalence class of $f$

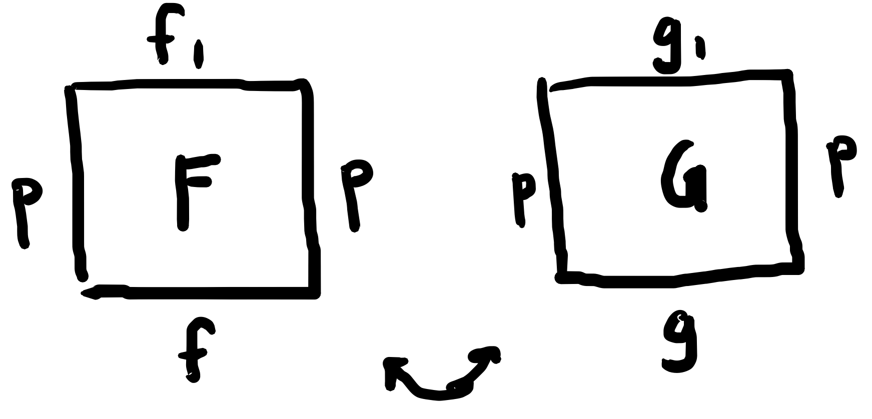
Concatenation:

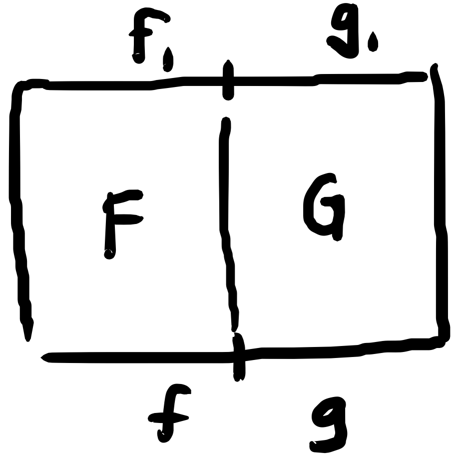
$$\pi_1(X,p) \times \pi_1(X,p) \longrightarrow \pi_1(X,p)$$

$$[f]\cdot[g] := [f * g]$$

$$(f*g)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1. \end{cases}$$

<u>Well-defined</u> : if $f_1 \sim f$ & $g_1 \sim g$ then $f_1 * g_1 \sim f * g$.




$$H(s,t) = \begin{cases} F(2s,t) & s \in [0,\tfrac{1}{2}] \\ G(2s-1,t) & s \in [\tfrac{1}{2},1] \end{cases}$$

$F * G$ ✓

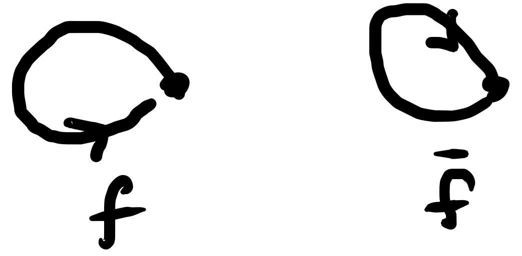
**Thm** Concatenation of homotopy classes satisfies: for $f, g, h$ any loops, $c \equiv p$ constant loop

(i) associativity $\quad ([f]\cdot[g])\cdot[h] = [f]\cdot([g]\cdot[h])$

(ii) identity $\quad [f]\cdot[c] = [f] = [c]\cdot[f]$

(iii) inverses $\quad [f][\bar{f}] = [c] = [\bar{f}][f]$

where $\bar{f}(s) = f(1-s)$ reverse of $f$.




**Cor** $\pi_1(X, p)$ is a group under concatenation

**Rmk** Thm isn't true on level of maps

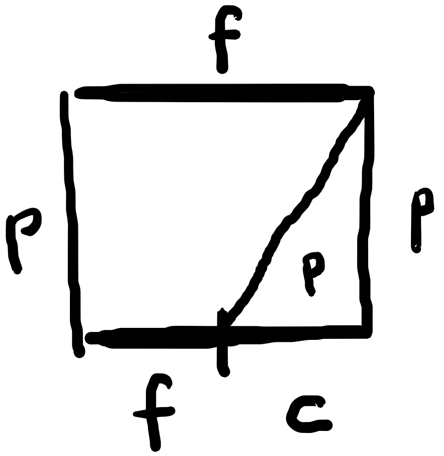
- need to consider homotopy classes.

$f * c \neq f$

$$(f * c)(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ p & 1/2 \leq s \leq 1. \end{cases} \neq f(s)$$

# Proof of Thm

- identity $\quad [f] * [c] = [f]$



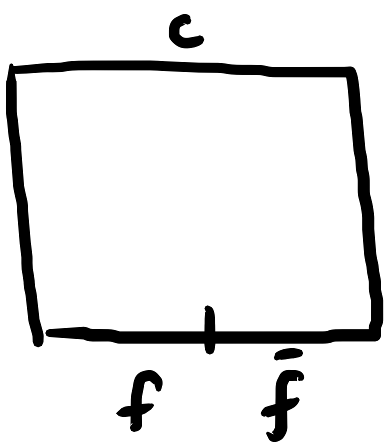

$$(f * c)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ p & 1/2 \le s \le 1 \end{cases}$$

$$[2(1-t) + t \cdot 1] = 2 - 2t + t = 2 - t$$

$$F(s,t) = \begin{cases} f((2-t) \cdot s) & 0 \le s \le \frac{1}{2-t} \\ p & \frac{1}{2-t} \le s \le 1 \end{cases}$$

✓

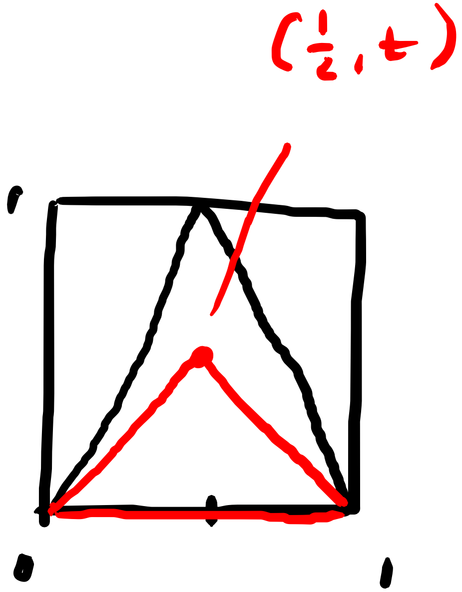
- Inverses




$$(f * \bar{f})(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ \bar{f}(2s-1) & 1/2 \le s \le 1 \end{cases}$$

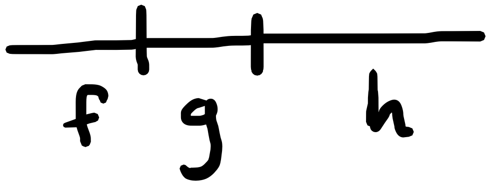
$$= \begin{cases} f(\underline{2s}) & 0 \le s \le 1/2 \\ f(\underline{2-2s}) & 1/2 \le s \le 1 \end{cases}$$

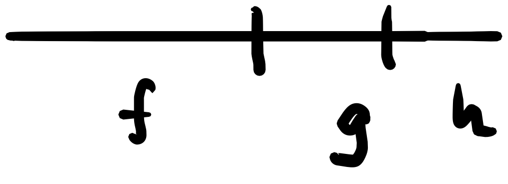
$$F(s,t) = \begin{cases} f(2ts) & 0 \le s \le 1/2 \\ f(2t(1-s)) & 1/2 \le s \le 1. \end{cases}$$

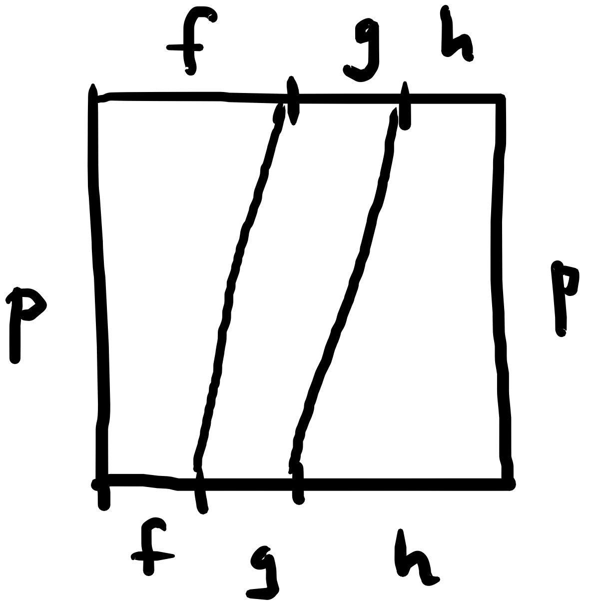


- associativity : you

$(f * g) * h$ $\qquad\qquad$ $f * (g * h)$

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