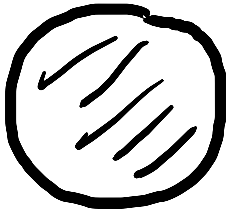
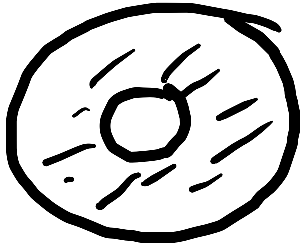
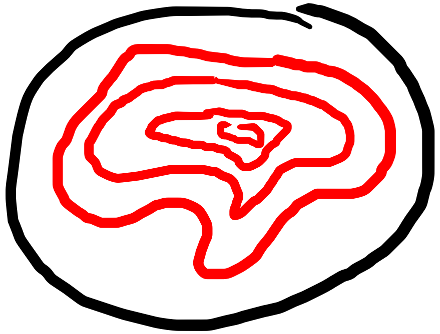
I. Fundamental group intro

<u>Ex.</u> What makes  different from  ?

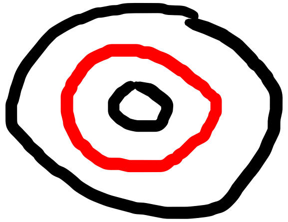
D^2 $\qquad\qquad$ $S^1 \times I$



- Answer: Euler number

  $$\chi(D^2) = 1 \neq 0 = \chi(S^1 \times I).$$

- Answer: properties of loops
  - on D^2 every loop can be deformed to a constant loop
  - on $S^1 \times I$ $\exists$ loops that can't be deformed to a constant



(requires proof)

We make "deform" precise with concept of homotopy.

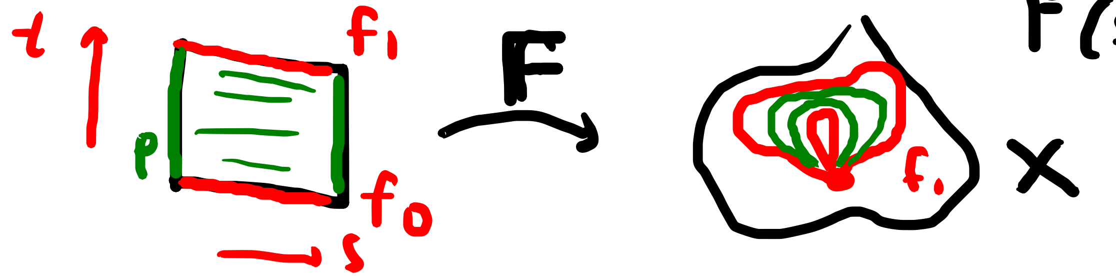
X space, $p \in X$ basepoint

<u>Defn</u> A <u>loop</u> based at p is a continuous map $f : [0,1] \to X$ with $f(0) = f(1) = p$.

We say $f_0, f_1 : [0,1] \to X$ are <u>homotopic</u> if $\exists$ continuous

$F : I \times I \longrightarrow X$ so that $F(s,0) = f_0$ $\quad F(0,t) = p$

(s,t) $\qquad F(s,1) = f_1$ $\quad F(1,t) = p$



<u>Ex</u>. Every loop in D^2 based at 0 is homotopic to a constant

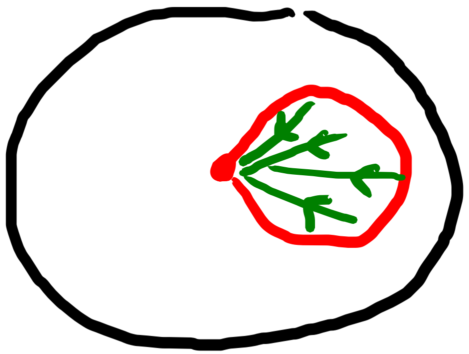
$f : [0,1] \to D^2$ any loop

$F(s,t) = t \cdot f(s)$.

$F(s,1) = f(s)$, $F(s,0) = 0$

$F(0,t) = t \cdot f(0) = 0$

$F(1,t) = t \cdot f(1) = 0$.



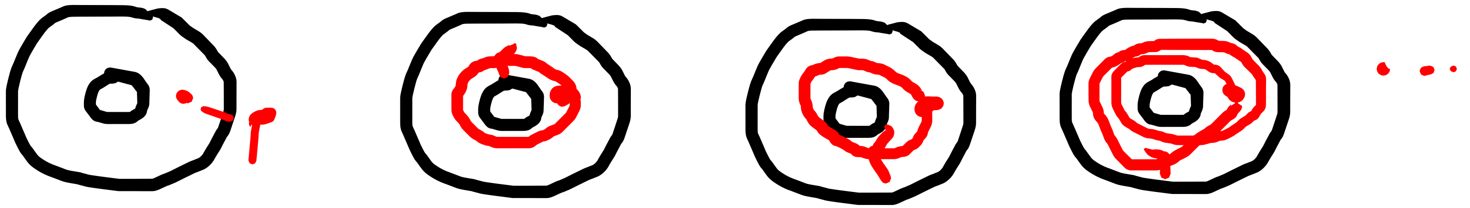
- Write $f \sim g$ if f, g homotopic
- This is an equivalence relation

($f \sim f$; $f \sim g \Rightarrow g \sim f$; $f \sim g$ and $g \sim h \Rightarrow f \sim h$)

- write $\pi_1(X, p) = \{\text{loops based at } p\}/\sim$

- eg on $\mathbb{D}^2$ there one equivalence class $\pi_1(\mathbb{D}^2, 0) = pt$

- on $S^1 \times I$ there are many equivalence classes



Later: $\pi_1(S^1 \times I, p) \cong \mathbb{Z}$

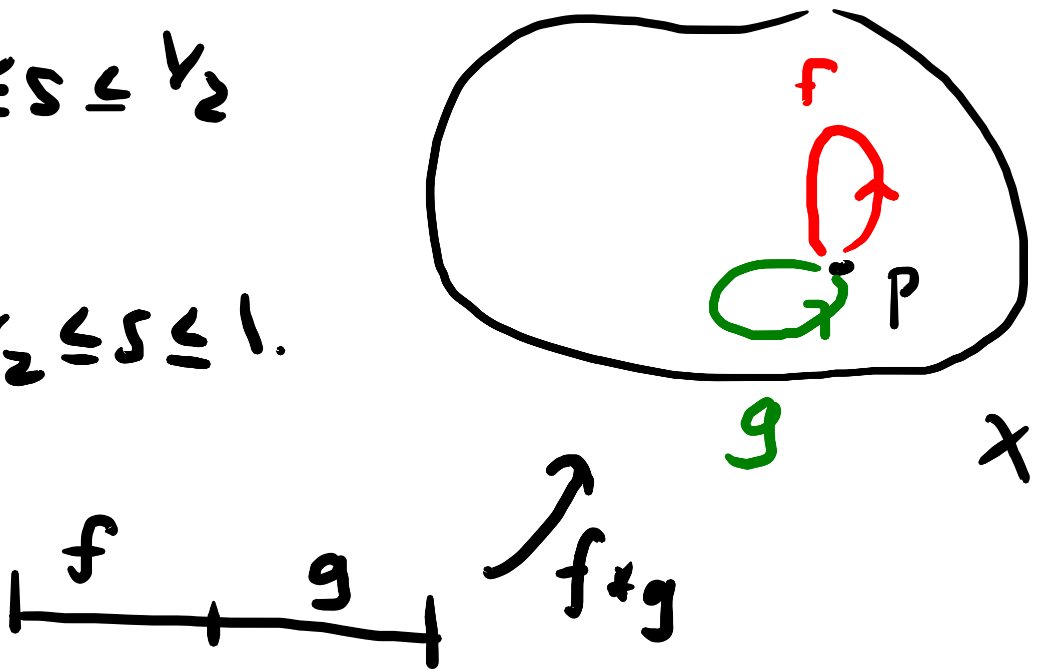
"winding number"

- $\pi_1(X,p)$ can be made into a <u>group</u> under concatenation:

given $f,g : [0,1] \longrightarrow X$

define $(f*g)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1. \end{cases}$

$f*g : [0,1] \to X$



Call $\pi_1(X,p)$ <u>fundamental group</u>

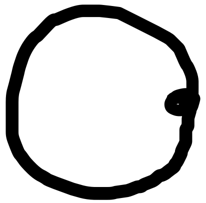
- Later $\pi_1(X,p)$ topological invariant, indep of choice of p if X connected

Eg $\quad \pi_1(\mathbb{D}^2) = 0 \qquad \pi_1(S^1 \times I) = \mathbb{Z}$

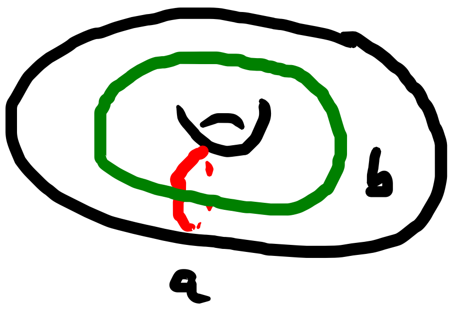
$\Rightarrow \quad D^2 \not\cong S^1 \times I.$

II. Intuitive computations

- $\pi_1(\mathbb{R}^2) = 0$
- $\pi_1(S^1) = \mathbb{Z}$



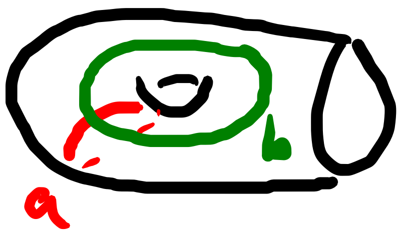
- $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$

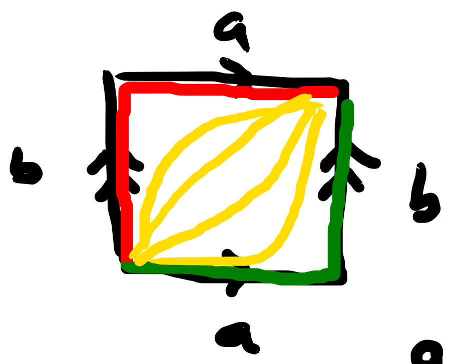


$\left(\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y) \right)$

- $\pi_1(T^2 \setminus D^2) \cong F_2 = \langle a, b \rangle$

  free group on two generators





$a * b = b * a$

- $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$

$a^2 = 1$

$\mathbb{R}P^2 = S^1 \cup_f D^2$