I. More orbit spaces

G group acting on space X.

$\mathcal{O}_x = \{ g\cdot x \mid g \in G \}$, $\quad X/G = \{ \mathcal{O}_x : x \in X \}$ orbit space

<u>Ex 1</u> Klein bottle $\quad X = \mathbb{R}^2$

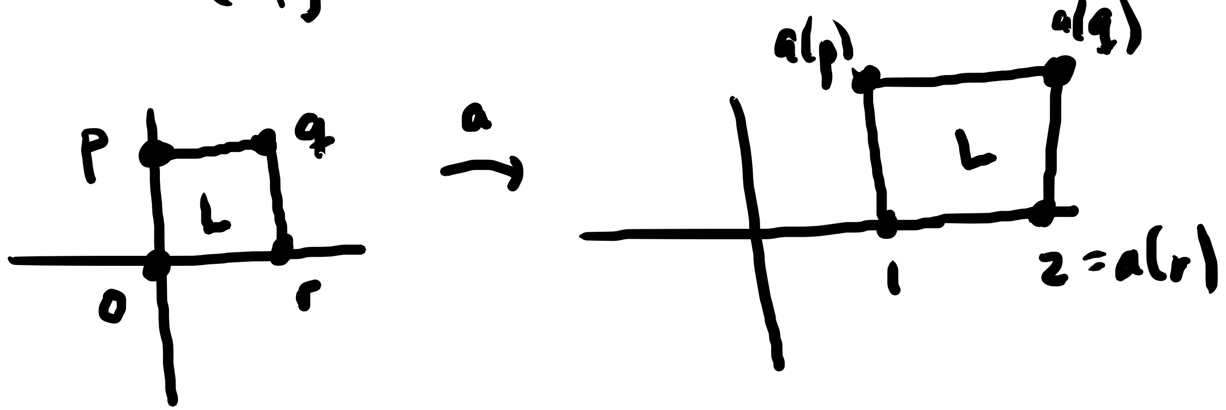
G group generated by two isometries of $\mathbb{R}^2$

$a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (x+1, y)$

$b : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ "glide reflection"

$(x,y) \longmapsto (-x, y+1)$

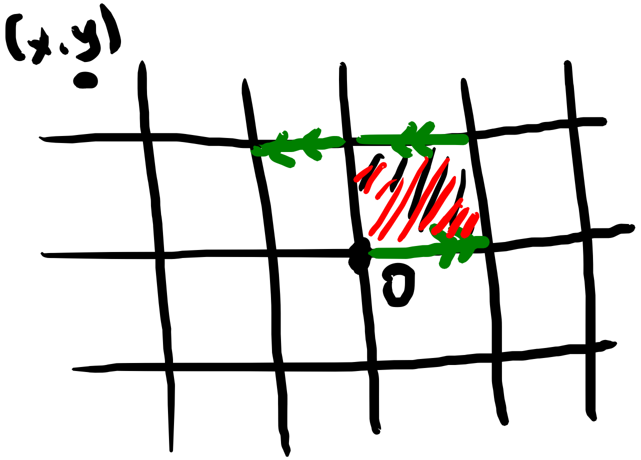


check: a, b don't commute (so $G \neq \mathbb{Z}^2$)

- $b^2(x,y) = b(-x, y+1) = (x, y+2)$

  does commute w/ a $\quad \langle a, b^2 \rangle \cong \mathbb{Z}^2 < G$

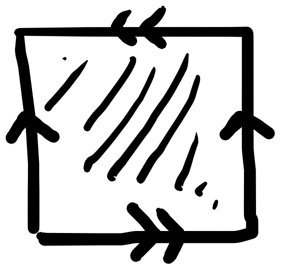
Determine X/G: observe that every $(x,y) \in \mathbb{R}^2$ in same orbit as a pt in $[0,1]^2$.



$\Rightarrow \exists$ surjection $[0,1]^2 \xrightarrow{q} X/G$

$X/G \cong$ associated partition of $[0,1]^2$

$= \{ q^{-1}(a) : a \in X/G \}$.

$X/G \cong$  $\cong K$

Klein

A general strategy for identifying a quotient space X/G

(1) Find a smallest possible subset $F \subset X$ that contains a point in each orbit (possibly more on boundary of F)

"fundamental domain"

(2) by (1) $\exists$ surj $F \xrightarrow{q} X/G$

so it suffices to determine partition of F induced by q

(ie how pts on boundary of F are glued).

Ex 2 Lens spaces

$X = S^3 = \{(z,w) \in \mathbb{C}^2 \mid z\bar{z} + w\bar{w} = 1\} \subset \mathbb{C}^2$

$(z = x+iy, \quad \bar{z} = x-iy, \quad z = re^{i\theta}, \quad \bar{z} = re^{-i\theta})$

$S^3 \ni \underbrace{(e^{i\theta}, 0)}_{S^1}, \underbrace{(0, e^{i\varphi})}_{S^1}, \left(\frac{1}{\sqrt{2}} e^{i\theta}, \frac{1}{\sqrt{2}} e^{i\varphi}\right)$

$G = \mathbb{Z}/p\mathbb{Z} = \langle t \mid t^p = 1 \rangle \quad (p \in \mathbb{N})$

Fix q s.t. $1 \leq q < p$ with $\gcd(p,q) = 1$.

define $t.(z,w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w)$.

$t^p.(z,w) = (e^{2\pi i} z, e^{2\pi i q} w) = (z,w)$

$\underline{G \to Top(S^3)}$

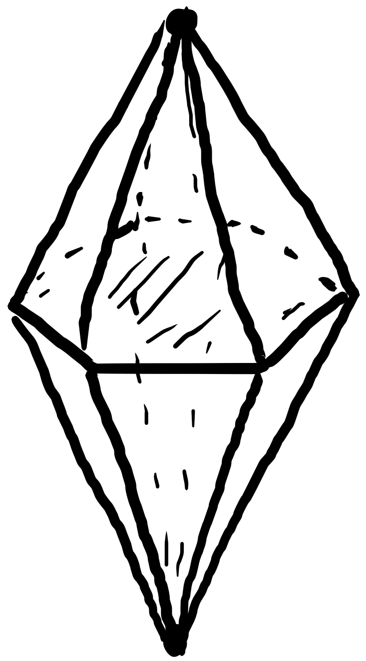
X/G is called a lens space denoted $L(p,q)$.

$\underline{\text{Ex}}$. $p=2$, $q=1$. $G=\mathbb{Z}/2\mathbb{Z} \curvearrowright S^3$

$t(z,w) = (e^{2\pi i/2} z, e^{2\pi i/2} w) = (-z,-w)$

antipodal map! $L(2,1) \cong \mathbb{R}P^3$

$\underline{\text{Ex}}$ $p=6$, $q=5$. We'll describe $L(6,5)$ as a quotient of



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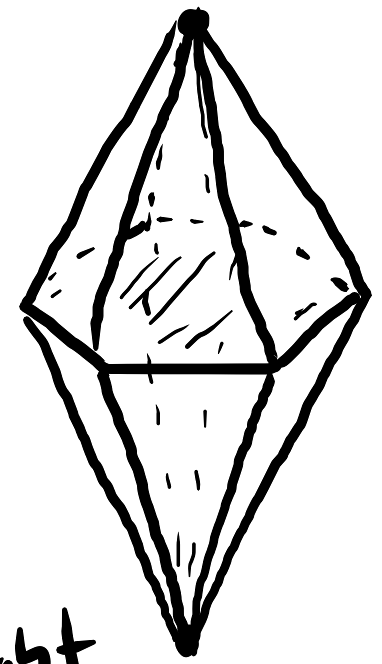
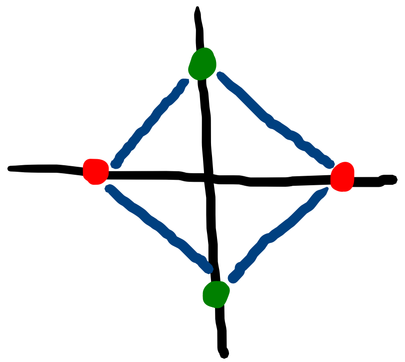


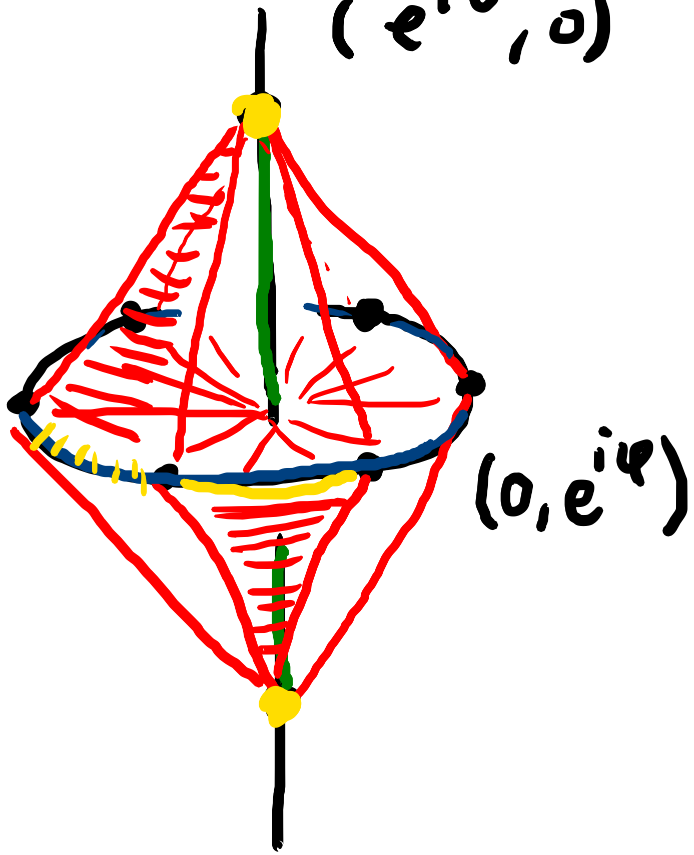
$S^3 = \underline{S^1} * \underline{S^1}$ join

$X * Y \subset \mathbb{R}^{n+m}$ $\qquad X \subset \mathbb{R}^n \quad Y \subset \mathbb{R}^m$

join every pt of X to every pt of Y by straight line

Ex. $S^0 * S^0 \cong S^1$ $\qquad S^p * S^q = S^{p+q+1}$



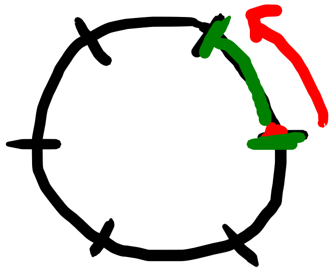


$\subset \mathbb{R}^3$

$S^3 = \mathbb{R}^3 \cup \{\infty\}$

G acts on $\{(e^{i\theta},0)\} \cong S^1$

as rotation by $2\pi/p$



G acts on $\{(0,e^{i\varphi})\} \cong S^1$

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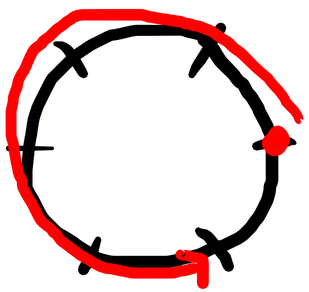
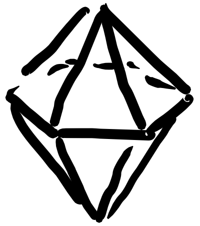


L(6,5) is quotient of  obtained by gluing bottom triangles to top triangles by a $\frac{2\pi q}{p}$ shift.

Rmk This construction / argument also gives a cell structure to $L(p,q)$.

Ex 3 $G = SO(n)$ $\quad X = S^{n-1} \subset \mathbb{R}^n$

X/G = single point (similar to HW 2 #5) "action is transitive"

boring — but there is an interesting partition of G

Consider $SO(n) \xrightarrow{q} S^{n-1}$ $\quad e_1 = (1, 0, \dots, 0)$

$A \longmapsto A \cdot e_1$ (= 1st column of A)

$G \curvearrowright X$ transitive $\Rightarrow$ q surjective. $\Rightarrow$ q quotient map

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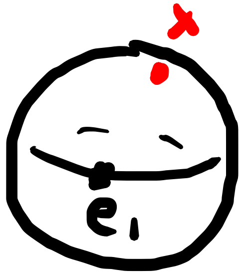
$\Rightarrow S^{n-1} \cong \{ q^{-1}(x) : x \in S^{n-1} \} =: \mathcal{P}$ partition of $SO(n)$

observe $A.e_1 = B.e_1 \Leftrightarrow \underline{B^{-1}A}\, e_1 = e_1 \Longleftrightarrow B^{-1}A = \left(\begin{array}{c|c} 1 & \\ \hline & C \end{array}\right)$

$C \in SO(n-1)$



$\Rightarrow$ Partition P of $SO(n)$

is the partition into cosets of $SO(n-1) < SO(n)$

$$\Rightarrow \quad SO(n)/SO(n-1) \cong S^{n-1}$$

__Cor__ $SO(n)$ connected for every n.

__Lemma__ (exercise, Armstrong 4.29) $G \curvearrowright X$

the quotient map $\pi: X \to X/G$ is open.

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the quotient map $\pi: X \longrightarrow X/G$ is open.

<u>Pf of Cor</u> (by induction on n)

<u>base case</u> $n=1$ $SO(1) = \{(a) : a^t a = 1, \ a = 1\} = \{1\}$.

($n=2 \to SO(2)$ connected by HW)

<u>Induction step</u>. Assume $SO(n-1)$ connected.

By contradiction, suppose $SO(n) = U \cup V$

Consider quotient map $SO(n) \xrightarrow{q} SO(n)/SO(n-1) \cong S^{n-1}$

By lemma q open map so $q(U), q(V)$ open.

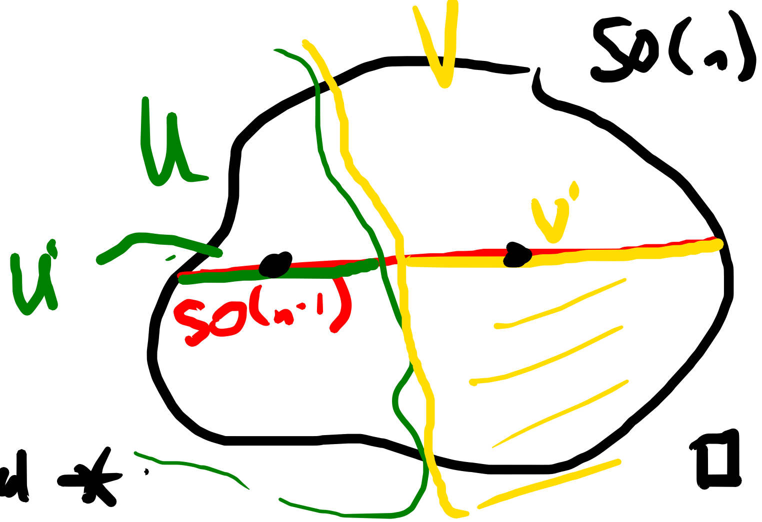
$S^{n-1} = q(U) \cup q(V)$. $\qquad e_1 \in q(U) \cap q(V)$

But S^{n-1} connected $\Rightarrow \exists\, x \in q(U) \cap q(V)$. wlog $x = e_1$

Consider $U' = U \cap SO(n-1) \quad V' = V \cap SO(n-1)$

Know U', V' disjoint, $U' \cup V' = SO(n-1)$.

nonempty: b/c $e_1 \in q(U)$ and $e_1 \in q(V)$. $\Rightarrow SO(n-1)$ disconnected $\divideontimes$



$\square$

$e_1 \in q(U)$

$\Rightarrow \exists\ A \in U$ s.t. $Ae_1 = e_1$

$\Rightarrow A \in SO(n-1) \cap U = U'$