I. Cell complexes

idea: build topological spaces from simple pieces

cell $\mathbb{D}^n = \{x \in \mathbb{R}^n : |x| \leq 1\} \supset \{|x| = 1\} = S^{n-1}$

<u>Construction</u> given space X, map $f : S^{n-1} \longrightarrow X$

consider partition on $X \sqcup \mathbb{D}^n$ $\quad \mathcal{P}$: $\{x, f(x)\}$ $\quad x \in S^{n-1}$

$\{a\}$ $\quad$ else

$a \in X \setminus f(S^{n-1})$

or $a \in \mathbb{D}^n \setminus S^{n-1}$

write $X \cup_f \mathbb{D}^n$ for $\mathcal{P}$ w/ quotient top.

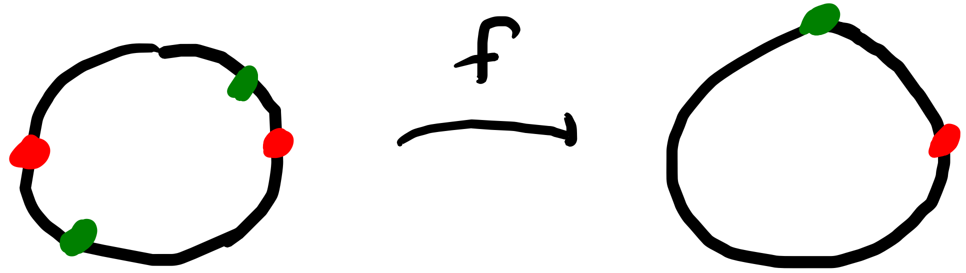
<u>Terminology</u> f called <u>attaching map</u>

$X \rightsquigarrow X \cup_f \mathbb{D}^n$ <u>cell attachment</u>

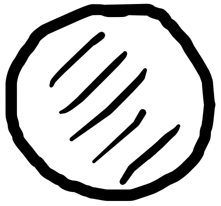
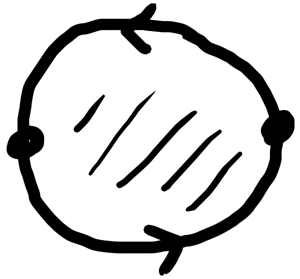
Ex. $X = S^1$ Fix $m \geq 1$ consider

$$f : S^1 \longrightarrow S^1 \qquad f(e^{i\theta}) = e^{im\theta} = (e^{i\theta})^m$$

eg $m=1$ $f = id$, $m = 2$ f is 2-to-1



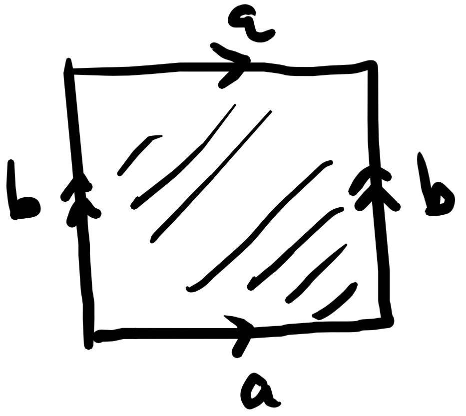
$$Z = X \cup_f D^2$$

$m=1$  , $m=2$  $= \mathbb{RP}^2$

$Z = D^2$

Ex. $X = S^1 \vee S^1$

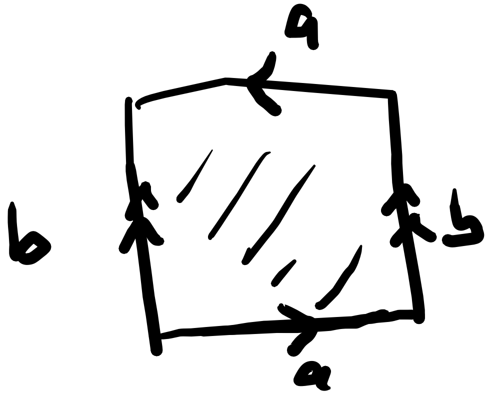
$S^1 \xrightarrow{f} X$ mapping along "$\underline{aba^{-1}b^{-1}}$"



$\cong T^2$

$T^2 \cong (S^1 \vee S^1) \cup_f \mathbb{D}^2$

if instead chose $abab^{-1}$



$=$ Klein bottle.

Defn A <u>cell complex</u> is a space obtained from a discrete set by attaching cells.

Thm (later) Every surface is a cell complex

Ex. $\mathbb{R}P^n = \{$ lines through 0 in $\mathbb{R}^{n+1} \}$

(as before view $\mathbb{R}P^n$ as a quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ with quotient topology.)

Thm $\mathbb{R}P^n$ is a cell complex

Explain case $n = 3$.

Write pts of $\mathbb{R}P^3$ as $[x : y : z : w]$

Ex. $\mathbb{R}P^n = \{$ lines through 0 in $\mathbb{R}^{n+1}\}$

Thm $\mathbb{R}P^n$ is a cell complex

Explain case $n=3$.

Write pts of $\mathbb{R}P^3$ as $[x:y:z:w]$

Observe $\mathbb{R}P^3 = A \sqcup B$ $\quad A = \{w=0\}$ $\quad B = \{w \neq 0\}$.

$A \cong \mathbb{R}P^2$. **Claim** $B \cong \mathbb{R}^3$ ($\cong$ interior of D^3)

Pf consider $B \longrightarrow \mathbb{R}^3$ $\quad$ with inverse $\quad \mathbb{R}^3 \longrightarrow B$

$[x:y:z:w] \longmapsto \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$ $\quad (x,y,z) \mapsto [x:y:z:1]$

(check: these maps are continuous)

III Orbit space

Recall A group action is a function

$G \times X \longrightarrow X$ s.t. • for $g \in G$ $x \longmapsto gx$ is a top equiv

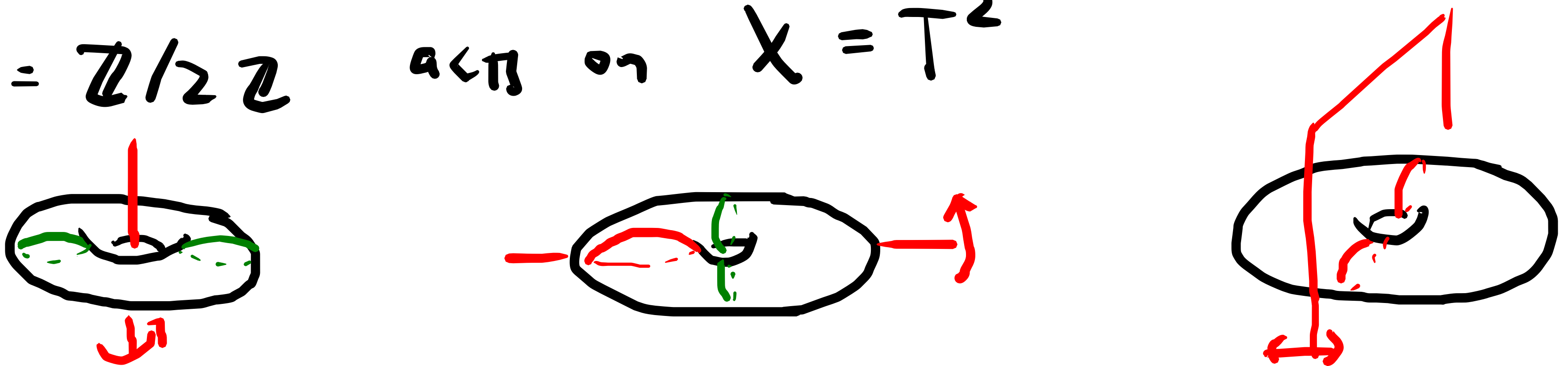
$(g, x) \longmapsto g \cdot x$ • $(gh) \cdot x = g \cdot (h \cdot x)$

Ex $G = \mathbb{Z}^2$ acting on $X = \mathbb{R}^2$ $(n, m) \cdot (x, y) = (x+n, y+m)$

Ex $G = \mathbb{Z}/2\mathbb{Z}$ acts on $X = S^2$ by antipodal map

$x \longmapsto -x$,

Ex. $G = \mathbb{Z}/2\mathbb{Z}$ acts on $X = T^2$

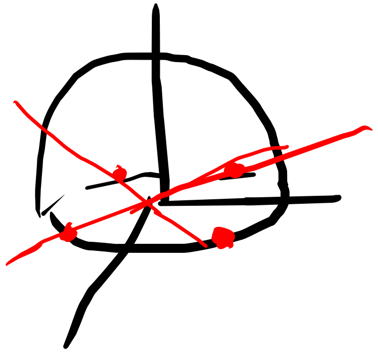


Claim $\mathbb{RP}^3 \cong \mathbb{RP}^2 \cup_f \mathbb{D}^3$ where

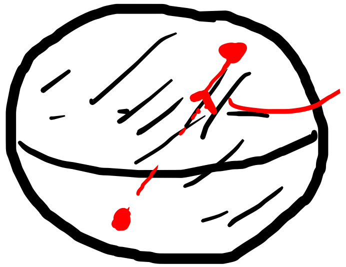
$f: S^2 \longrightarrow \mathbb{RP}^2$ is quotient map (last time)

Idea Fix $(x,y,z) \in \mathbb{R}^3$ look at image of (tx,ty,tz) in B as $t \to \pm\infty$.

$$(tx,ty,tz) \longmapsto [tx : ty : tz : 1] \in B$$
$$= [x : y : z : \tfrac{1}{t}]$$



converges to $[x:y:z:0]$ as $t \to \pm\infty$.
$\in A = \mathbb{RP}^2$



B

Rmk This description of $\mathbb{RP}^3$ as quotient of $\mathbb{D}^3$ is analogous to $\mathbb{RP}^2 = \mathbb{D}^2/\sim$

Defn G acts on X. For $x \in X$ the **orbit** of x is $\mathcal{O}_x = \{ gx \mid g \in G \}$

Note that orbits are either disjoint or equal.
if $\mathcal{O}_x \cap \mathcal{O}_y \neq \emptyset$ then $\mathcal{O}_x = \mathcal{O}_y$. (exercise)

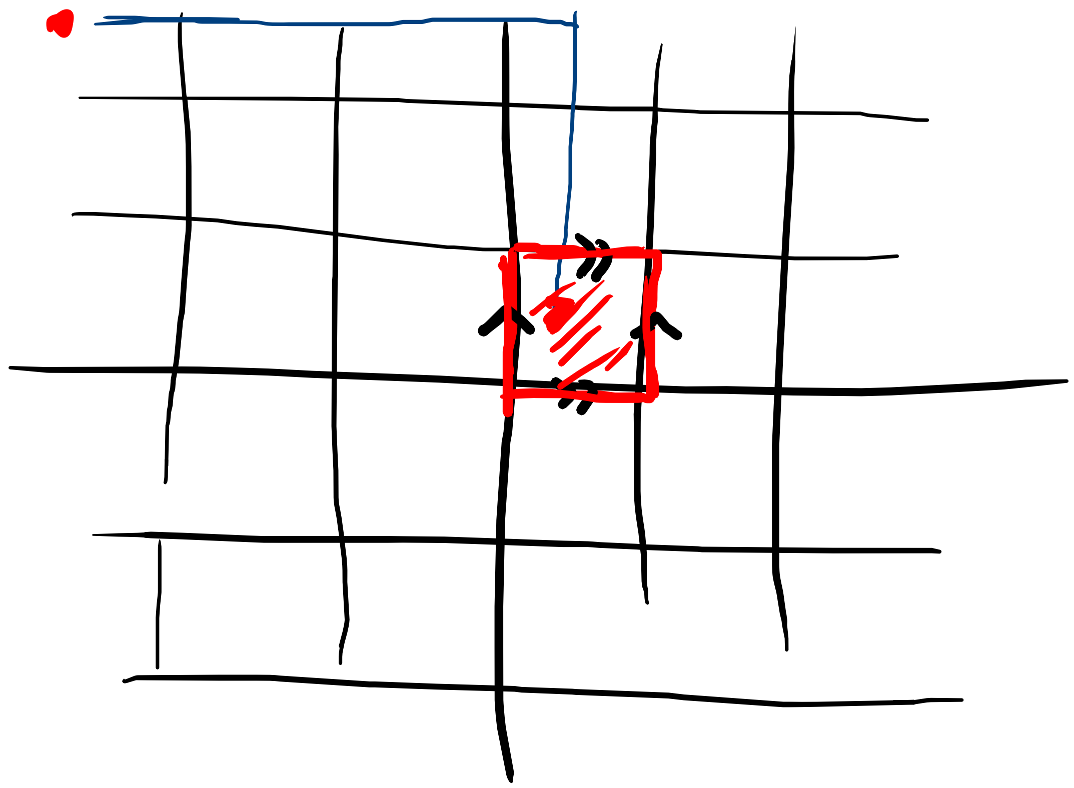
$\Longrightarrow$ $\mathcal{P} = \{ \mathcal{O}_x : x \in X \}$ partition of X.

write $\underbrace{X/G}_{\text{orbit space}}$ for this partition w/ quotient top.

Ex $G = \mathbb{Z}^2$ acting on $X = \mathbb{R}^2$

$(n,m) \cdot (x,y) = (x+n, y+m)$

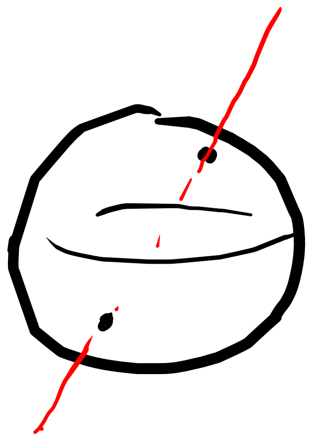
observe every orbit of form $\mathcal{O}_{(x,y)}$ where $(x,y) \in [0,1]^2$



$X/G \cong T^2$

(2) $G = \mathbb{Z}/2\mathbb{Z}$ acting on $X = S^2$ antipodal.

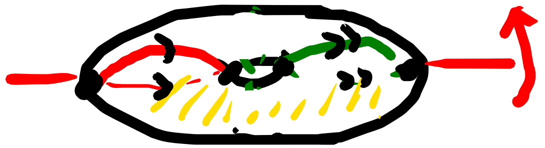
orbits $\mathcal{O}_x = \{x, -x\}$.

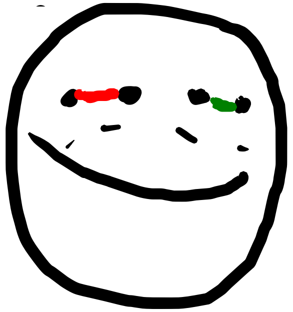


$$X/G \cong \mathbb{RP}^2$$

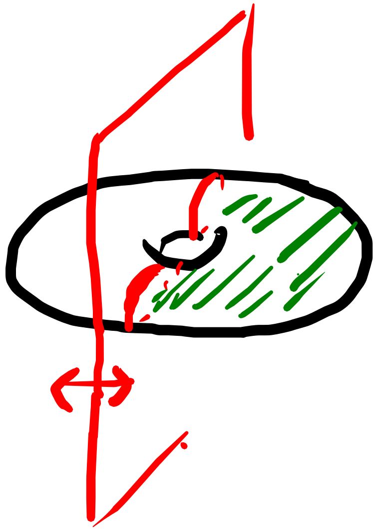
Ex. $G = \mathbb{Z}/2\mathbb{Z}$ act on $X = T^2$

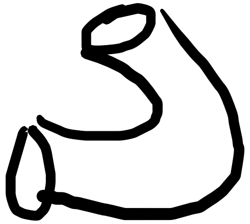
$X/G = T^2$



$X/G =$ 

S^2



$X/G =$ 

$S^1 \times [0,1]$.