I. Connectedness

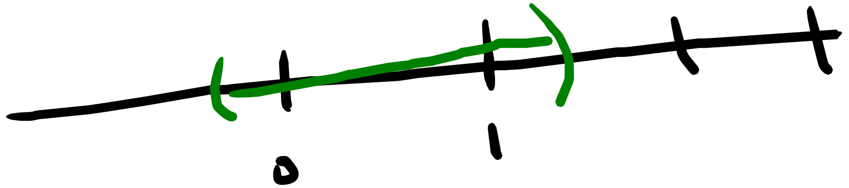
<u>Defn</u> X <u>disconnected</u> if can write

$X = U \cup V$ U, V nonempty, disjoint, open

otherwise X is <u>connected</u>   (whenever $X = U \cup V$ U, V nonempty then either U contains limit point of V or conversely)

<u>Examples</u>

- $X = [0,1] \cup [2,3] \subset \mathbb{R}$ $U = [0,1]$ $V = [2,3]$



$X = (0,1) \cup (2,3)$

disconnected

- $X = \mathbb{Q}$ disconnected

  $X = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$

- $\mathbb{R}$ with discrete topology is disconnected

  $\mathbb{R} = \{0\} \cup \mathbb{R} \setminus \{0\}$

- $\mathbb{R}$ with indiscrete topology is connected

  ($\nexists$ disjoint nonempty open sets)

Prop $\mathbb{R}$ w/ standard topology
is connected

Rmk connected is a topological property
$(X \cong Y \rightarrow$ X connected iff Y connected$)$

by prop open intervals $(0,1)$ also connected

Recall LUB property of $\mathbb{R}$: any nonempty $A \subset \mathbb{R}$
that's bounded above has a least upper bound.

Rmk Not true if replace $\mathbb{R}$ with $\mathbb{Q}$. eg $A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$

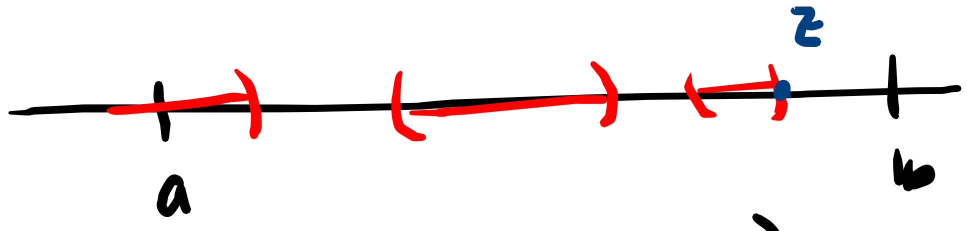
Prop $\mathbb{R}$ w/ standard topology is connected

Proof Suppose for contradiction $\mathbb{R}$ disconnected

$\mathbb{R} = U \cup V$. Take $a \in U$, $b \in V$

Consider $A = U \cap [a,b]$ nonempty, bounded above

So A has a least upper bound z



Case 1 $z \in U$. Then z is a limit point of V ($z^{\notin V}$ either exterior pt or limit of V) $\square$

↳ can't happen b/c z upperbound for U.

Case 2 $z \in V$. similar. z limit point of U

Rmk same argument shows $[0,1]$ connected

II. Connectivity & Continuity.

Lemma X disconnected $\iff$ $\exists$ continuous surjection $f: X \to \{0,1\}$ ← discrete top.

Proof ($\Rightarrow$) $X = U \cup V$

define $f: X \to \{0,1\}$ $f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$

($\Leftarrow$) Given $f: X \to \{0,1\}$ define $U = f^{-1}(0)$ $V = f^{-1}(1)$.

$\square$

Example $GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}$

Claim $GL_2\mathbb{R}$ disconnected

Consider $f : GL_2\mathbb{R} \longrightarrow \{-1, 1\}$

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} 1 & ad - bc > 0 \\ -1 & ad - bc < 0 \end{cases}$$

$f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$f\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1$

$$f : GL_2\mathbb{R} \xrightarrow{\det} \mathbb{R}^\times \xrightarrow{\text{sign}} \{-1, 1\}$$

Rmk For $SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$ this argument doesn't work. In fact $SL_2\mathbb{R}$ connected (HW)

Lemma X connected, $f: X \to Y$ continuous, surjective

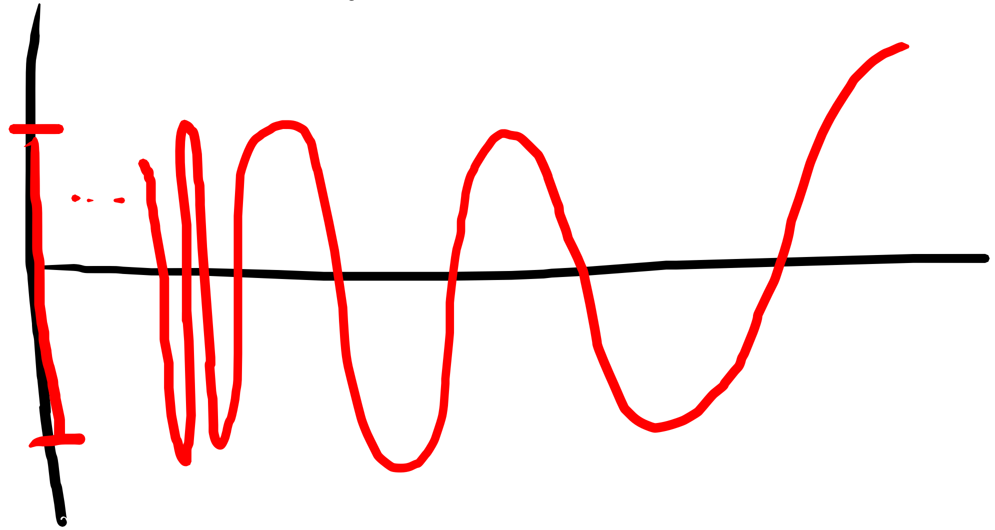
Then Y connected.

Ex $[0,1]$ connected $\Rightarrow$ S^1 connected

Proof of lemma: if Y disconnected. $Y = U \cup V$

Then $X = f^{-1}(U) \cup f^{-1}(V)$ $\Rightarrow$ X disconnected. $\square$

Ex Topologist Sine curve



Claim $X = \underbrace{(\{0\} \times [-1,1])}_{I_1} \cup \underbrace{\{(x, \sin\frac{1}{x}) : x > 0\}}_{I_2}$ is connected!

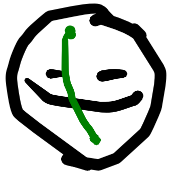
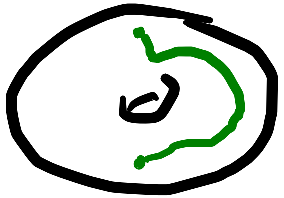
Sketch Suppose $X = U \cup V$. $I_1 \cong [0,1]$ connected

$\Rightarrow I_1 \subset U$ or $I_1 \subset V$.

Any open set in $\mathbb{R}^2$ containing I_1 Same for $I_2 \cong (0,\infty) \cong \mathbb{R}$

Contains points of I_2. $\Rightarrow U, V$ not both nonempty.

Rmk in practice it can be hard to show a space is connected using the definition

eg $\mathbb{R}^n$,  , 

Next: path connectedness