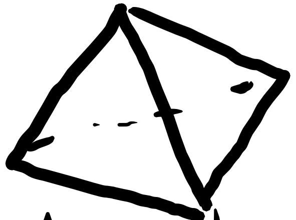
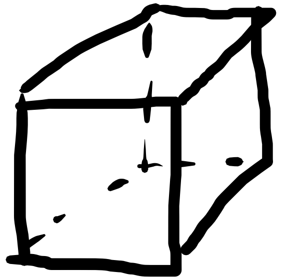
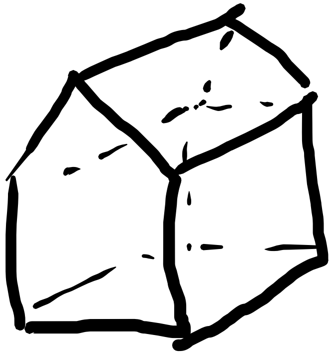
I. Euler's Theorem

A **polyhedron** is a union of polygons in $\mathbb{R}^3$

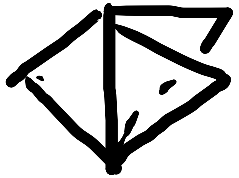


tetrahedron



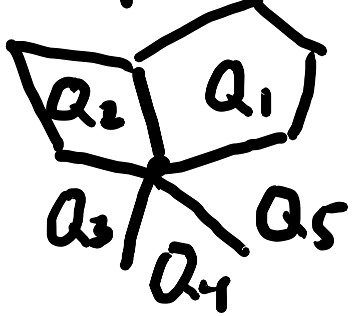


s.t. • each edge is in exactly 2 polygons

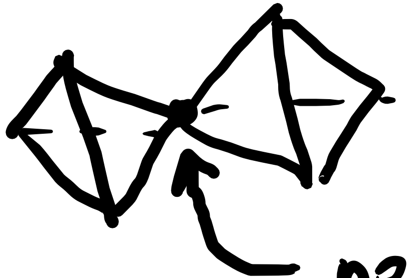


not allowed

• local picture around vertex



Q_i, Q_{i+1} share an edge

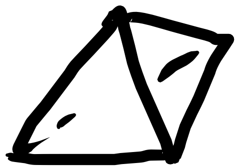


not allowed

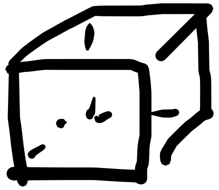
For polyhedron P denote V, E, F number of vertices, edges, faces

Curious fact for the examples above

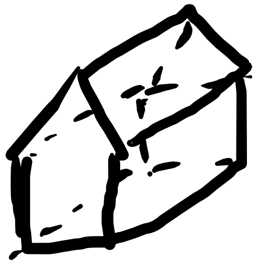
$V - E + F = 2$ (why not $V + E + F$?)



$V - E + F = 4 - 6 + 4 = 2$



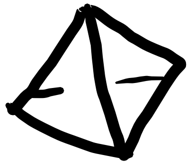
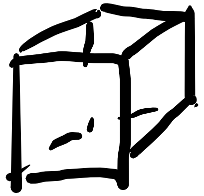
$V - E + F = 8 - 12 + 6 = 2$



$V - E + F = 10 - 15 + 7 = 2$

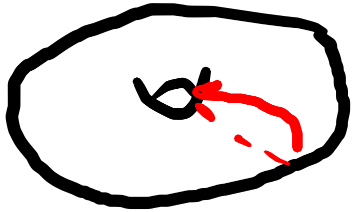
Q: is $V - E + F = 2$ for every polyhedron?

Q: is $V-E+F=2$ for every polyhedron?

No eg $P =$  $\sqcup$  $V-E+F=4$

Another example $P=$  ← torus polyhedron

$V-E+F = 16-32+16 = 0$

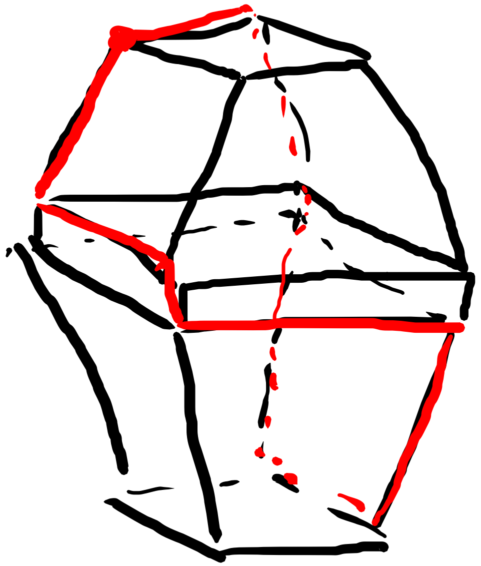


Euler's Thm (1700s) P polyhedron

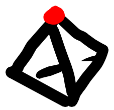
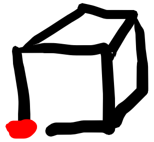
Suppose that

(i) any two vertices of P are connected by sequence of edges (connected)

(ii) any polygonal loop separates P into two pieces. (Simply connected)



Then $V - E + F = 2$ for P.

Rmk
- $P =$  $\sqcup$  fails (i)
- $P =$ torus polyhedron fails (ii)

- $V - E + F = \chi(P)$ Euler number / characteristic of P

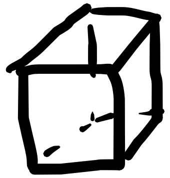
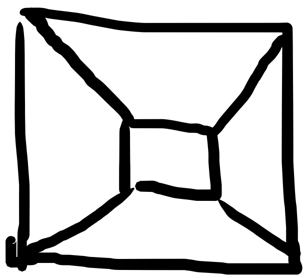
II. Graphs

Graph = "1-dimensional polyhedron"

vertices (finite set)

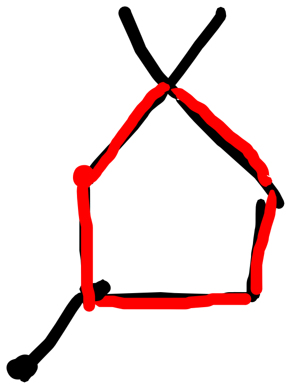
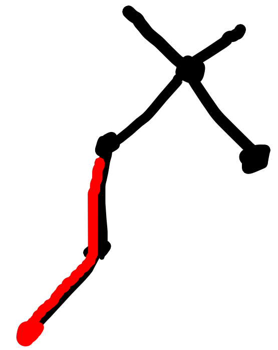
edges: connect distinct pairs of vertices

<u>Ex</u> P polyhedron, $G = P \setminus$ faces

$P =$  $G =$ 

- G is <u>connected</u> if any two vertices differ by a sequence of edges

A connected graph G is a <u>tree</u> if removing any edge disconnects G.

<u>Ex</u>  is not a tree b/c $G \setminus e =$  is connected

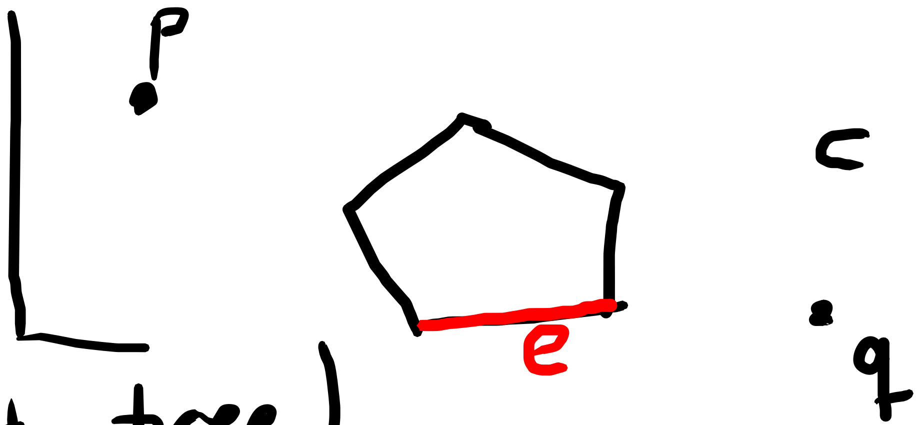
<u>Lemma</u> G connected graph. Then

G is a tree $\iff$ any loop in G backtracks.

A connected graph G is a <u>tree</u> if removing any edge disconnects G.

<u>Lemma</u> G connected graph. Then
G is a tree $\iff$ any loop in G backtracks.

<u>Proof</u> ($\Rightarrow$) Prove the contrapositive:
G has nonbacktracking loop:



$\subset G$

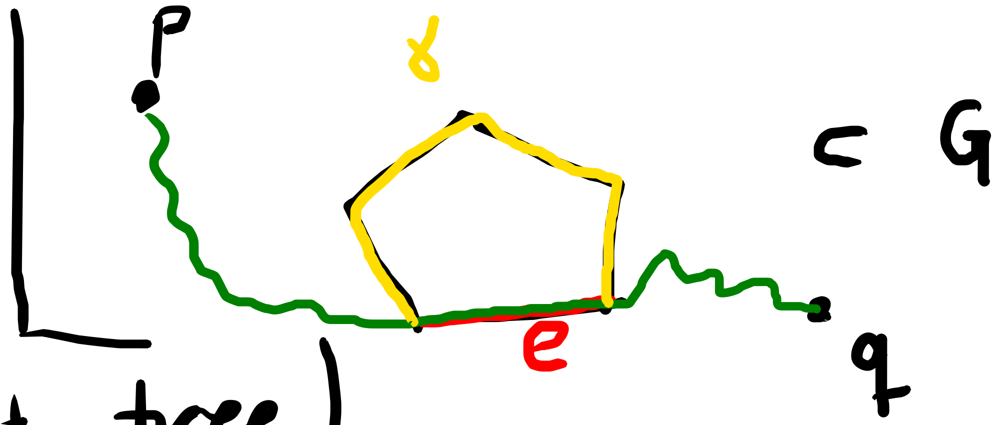
<u>Claim</u> $G \setminus e$ is connected ($\Rightarrow$ G not tree.)

<u>Pf of Claim</u> Fix p, q vertices of $G \setminus e$. Want sequence of edges in $G \setminus e$ connecting them.

<u>Lemma</u> G connected graph. Then

G is a tree $\Leftrightarrow$ any loop in G backtracks.

<u>Proof</u> ($\Rightarrow$) Prove the contrapositive:



G has nonbacktracking loop:

<u>Claim</u> $G \setminus e$ is connected ($\Rightarrow$ G not tree.)

<u>Pf of Claim</u> Fix p, q vertices of $G \setminus e$. Want sequence of edges in $G \setminus e$ connecting them.

Know $\exists$ sequence in G (G connected).

<u>Case 1</u> This sequence doesn't contain e — done

Case 2 sequence contains e — replace e by γ. ✓

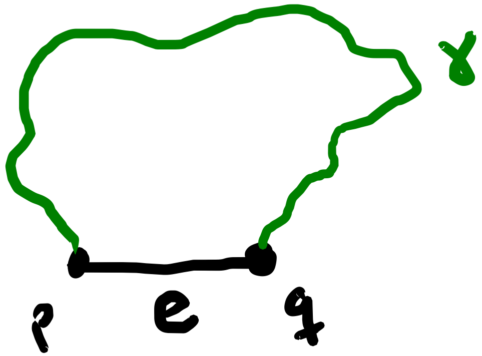
Lemma G connected graph. Then

G is a tree $\iff$ any loop in G backtracks.

Proof $(\Leftarrow)$ prove contrapositive:

G not a tree $\Rightarrow$ $\exists$ edge e s.t. $G \setminus e$ connect.

Let p, q vertices of e. $G \setminus e$ connect $\Rightarrow$ $\exists$ path γ nonbacking in $G \setminus e$ from p to q.



Then $\gamma \cup e$ is a nonbacktracking loop in G.

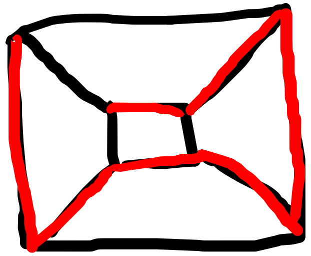
$\hookrightarrow$ $\exists$ nonbacking loop in G.

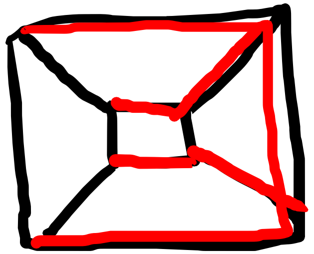
$\square$

Exercise (HW1) For any graph G $\exists$ tree $T \subset G$ that contain every vertex of G.

(T usually not unique)

eg





Lemma T tree. V vertices, E edges.

Then $V - E = 1$.

Proof induct on number of edges E

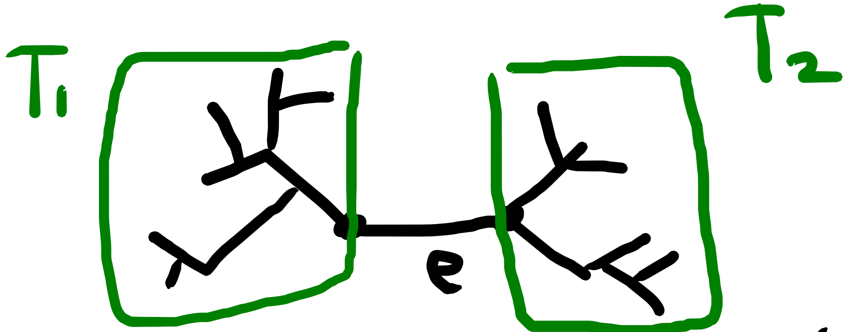
base case $E = 1$ $\quad T = \bullet\!\!-\!\!-\!\!\bullet \qquad V - E = 2 - 1 = 1.$ ✓

Fix $E > 1$.

IH Assume lemma holds for trees w/ $< E$ edges.

Fix T with E edges. Pick any edge $e \in T$

$T \setminus e = T_1 \cup T_2$



$$V_T - E_T = V_{T_1} + V_{T_2} - (E_{T_1} + E_{T_2} + 1) = \overset{1}{\overset{\shortparallel}{(V_{T_1} - E_{T_1})}} + \overset{1}{\overset{\shortparallel}{(V_{T_2} - E_{T_2})}} - 1$$

$$= 1. \qquad \square$$