Homework 9

Math 141

Due Tuesday, November 24, 2020 by 5pm

Topics covered: simplicial approximation, fundamental group, Sprouts, surfaces

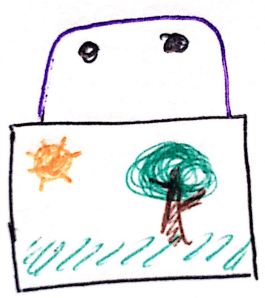
Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *Let $|K| = |L| = [0,1]$ with K having vertices at $0, \frac{1}{3}, 1$ and L at $0, \frac{2}{3}, 1$. Let $f : |K| \to |L|$ be defined by $f(x) = x^2$. Show that there is no simplicial approximation to $f : |K^1| \to |L|$. Give an example of a simplicial approximation to $f : |K^2| \to |L|$ that is different from the one given in Figure 6.9 in Armstrong.*

Solution. □

Problem 2. *Consider a picture frame hanging on a wall, supported by two nails and a string as shown below.*



Observe that the picture will remain hanging even if a single nail is removed. Find a way to hang the string on the nails so that the picture will fall if either nail is removed (the string must be attached to the frame in the same way). Draw a picture.[1]

Solution. □

Problem 3. *In the setup of the previous problem, suppose now that there are three nails.*

(a) *Find a way to hang the string so that it falls if any two nails are pulled from the wall, but not before.*

(b) *Find a way to hang the string so that it falls if any nail is pulled from the wall.*[2]

Express your answer using algebra instead of a drawing.

Solution. □

Problem 4. *Fix $n \geq 1$. Suppose G is a connected graph in the plane with $3n-1$ vertices and $4n-2$ edges. Assume further that every vertex has degree 2 or 3.* [3]

(a) *Determine the number of vertices of degree 2.* [4]

(b) *Think of G as a graph in the 2-sphere. Determine the number of regions of $S^2 \setminus G$.* [5]

[1] Hint: this is a problem about winding numbers.

[2] Hint: commutators.

[3] The degree of a vertex is the number of edges incident to it. A self-loop counts as 2.

[4] Hint: If there are E edges and V vertices, then there is a formula $2E = \sum_{v \in V} deg(v)$. Set up a system of linear equations and solve.

[5] Use's Euler's formula $2 = V - E + F$.

Solution. □

Problem 5. *Prove that a game of Sprouts with n initial vertices cannot end before $2n$ turns.*[6]

Solution. □

Problem 6 (Armstrong 7.22)**.** *Are the surfaces shown below topologically equivalent?*

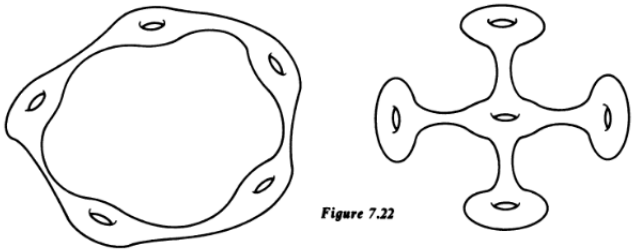


Figure 7.22

Solution. □

Problem 7 (Armstrong 7.33)**.** *Identify the two surfaces shown below.*

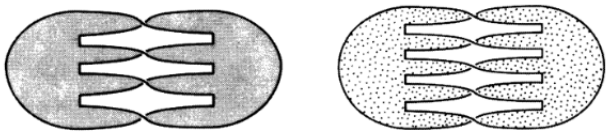


Figure 7.24

Can you suggest a general result from these two pictures?

Solution. □

This is the end of the homework ☺

[6]Suggestion: this can be done using an induction argument. The previous problem treats a special case.