# Homework 7

Math 141

Due November 6, 2020 by 5pm

Topics covered: fundamental group of the circle, Brouwer fixed point

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

**Problem 1** (Armstrong 5.21)**.** *Describe the homomorphism* $f_* : \pi_1(S^1) \to \pi_1(S^1)$ *induced by the following maps.*

*(a) The antipodal map* $f(e^{i\theta}) = e^{i(\theta+\pi)}$.

*(b) The map* $f(e^{i\theta}) = \begin{cases} e^{i\theta} & 0 \le \theta \le \pi \\ e^{i(2\pi-\theta)} & \pi \le \theta \le 2\pi \end{cases}$.

*Solution.* □

**Problem 2** (Armstrong 5.8)**.** *Consider the annulus*

$$A = \{(r, \theta) : 1 \le r \le 2, 0 \le \theta \le 2\pi\}.$$

*Let* $h : A \to A$ *be the topological equivalence defined by*

$$h(r, \theta) = (r, \theta + 2\pi(r - 1)).$$

*Show that* $h$ *is homotopic to the identity.*

*Solution.* □

**Problem 3** (Armstrong 5.23)**.** *Let* $A$ *be the annulus and* $h : A \to A$ *as defined above. Show that there is no homotopy from* $h$ *to the identity that is constant (equal to the identity) on the boundary of* $A$*. Do this as follows: consider paths* $\alpha, \beta : [0, 1] \to A$ *defined by*

$$\alpha(s) = (s + 1, 0) \quad \textit{and} \quad \beta(s) = h \circ \alpha(s).$$

*Show that if* $h$ *is homotopic to the identity through a homotopy that is constant on the boundary of* $A$*, then* $\bar{\alpha} * \beta$ *is homotopic to the constant loop. Derive a contradiction.*

*Solution.* □

**Problem 4** (Armstrong 5.30)**.** *Give detailed proof that the Möbius band is homotopy equivalent to the circle.*

*Solution.* □

**Problem 5.**

*(a) Using the techniques from class, can you prove that every map* $D^n \to D^n$ *has a fixed point? Explain.*

*(b) Explain how the 1-dimensional version of the Brouwer fixed point theorem follows from the Intermediate Value Theorem (c.f. HW4).*

*Solution.* □

**Problem 6.**

(a) *Show there is no way to tile an equilateral triangle $T$ with side length 6 with unit length hexiamonds, using none more than once.*[1]

(b) *(Extra credit – 5 points) Make physical hexiamonds (e.g. out of cardboard). Give a tiling of the $6 \times 6$ parallelogram by hexiamonds (each hexiamond will be used* exactly *once!). Submit a picture of your solution.*

*Solution.* □

[1]Hint: Color the triangles of each hexiamond with two colors so that adjacent triangles have different colors. Do the same with the triangles of $T$ (how many are there?). What do you notice?