Homework 6

Math 141

Due October 30, 2020 by 5pm

Topics covered: homotopy, fundamental group

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *Prove that $\mathbb{R}^{n+1}$ is not a union of finitely many planes through the origin. Conclude that S^n is not a union of finitely many great circle arcs.*

Solution. □

Problem 2 (Armstrong 5.7)**.** *Show that $f : X \to Y$ is homotopic to a constant*[1] *if and only if f extends to a map $\hat{f} : C(X) \to Y$, where $C(X) = X \times [0,1]/X \times \{1\}$ is the cone on X.*

Solution. □

Problem 3. *We say that spaces X, Y are homotopy equivalent if there exists maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. Show that homotopy equivalence of spaces is an equivalence relation.*[2]

Solution. □

Problem 4 (Armstrong 5.14)**.** *Let $\mathbb{R}^3_+$ denote the set of points $(x, y, z) \in \mathbb{R}^3$ with $z \geq 0$. Let*

$$W = \{(x, y, z) : y = 0, 0 \leq z \leq 1\}.$$

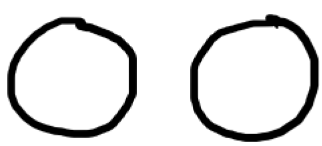
Prove that $\mathbb{R}^3_+ \setminus W$ is simply connected.

Solution. □

Problem 5. *Let X be the complement of the Hopf link in $\mathbb{R}^3$. Let a, b be the obvious elements of the fundamental group that go around the separate loops.*

(a) Is $aba^{-1}b^{-1}$ trivial in $\pi_1(X)$? (I.e. do a, b commute?)

(b) Repeat this exercise for Y the complement of the two-component unlink in $\mathbb{R}^3$. [3]



Solution. □

[1] We only defined homotopy of loops, so you should first figure formulate a more general definition of homotopic maps $f, g : X \to Y$. (See Armstrong §5.1.)

[2] We will see examples of this soon. This exercise can be solved formally by following the definitions.

[3] Look up the Borromean rings. How is that relevant here?