Homework 4

Math 141

Due October 9, 2020 by 5pm

Topics covered: compactness, connectedness, least upper bound property

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *Let X be compact and Y Hausdorff, and suppose that $f : X \to Y$ is a continuous bijection. Show that f is a topological equivalence.*[1]

Solution. □

Problem 2. *Follow the following outline to give a proof that $[0, 1]$ is compact (different from the one given in lecture). Let $\mathcal{U}$ be an open cover of $[0, 1]$, and consider the set*

$$A = \{x \in [0, 1] : \text{ there is a finite subcover of } \mathcal{U} \text{ that covers } [0, x]\}.$$

Let z be the least upper bound of A (why does it exist?). Prove that $z = 1$ and conclude.[2]

Solution. □

Problem 3.

(a) Let $A \subset \mathbb{R}$ be a nonempty compact subset. Prove that A has a maximal element, i.e. there exists $b \in A$ so that $a \leq b$ for every $a \in A$.

(b) Prove the maximum value theorem: if X is compact, and $f : X \to \mathbb{R}$ is continuous, then there exists $y \in X$ so that $f(x) \leq f(y)$ for all $x \in X$.

Solution. □

Problem 4 (Armstrong 3.33)**.** *Use connectedness to give a short proof of the following fact from analysis (known as the Intermediate Value Theorem).*[3] *Fix a continuous map $f : [0, 1] \to \mathbb{R}$. Show that if $f(0) < 0$ and $f(1) > 0$, then there exists $c \in (0, 1)$ so that $f(c) = 0$.*

Solution. □

Problem 5.

(a) Let $O(2)$ be the group of 2×2 matrices A so that $A^t A = I$. Let $SO(2) \subset O(2)$ be the subset of matrices with determinant 1. Is $O(2)$ connected? What about $SO(2)$?

(b) (Extra credit) Show that $SL_2(\mathbb{R})$ is (path) connected.[4]

Solution. □

Problem 6. *Let S be a subset of $\mathbb{R}$.*

(a) Give an algebraic proof that the maximum number of subsets you can obtain from S by the operations closure and complement is at most 14. [5]

[1] Frame the problem in terms of a property of f. You may want to consider equivalent formulations. Your solution should be short.

[2] Are you done immediately after showing $z = 1$? Be careful.

[3] The standard proof in analysis uses the least upper bound property. The proof you give will have the LUB property in the background.

[4] Look up the polar decomposition of a matrix.

[5] Hint: If c denotes complement and b denotes closure, then there are two easy relations: $c^2 = \text{id}$ and $b^2 = b$. There is one more relation needed to solve this problem.

(b) (Extra credit) Find a set S that produces 14 (give proof!). [6]

Solution. □

[6]Hint: Find a set that has the following features: an isolated point in the set, an isolated point in the complement, limit points in the set, limit points in the complement, only the rational numbers in some interval.