# Homework 3

Math 141

Due October 2, 2020 by 5pm

Topics covered: Topological spaces, continuity, compactness

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

**Problem 1.** *Classify the Hausdorff topologies on the set* $X = \{a, b, c, d\}$.

*Solution.* □

**Problem 2.** *Recall that in class we showed that if* $X \subset \mathbb{R}^n$ *is compact, then* $X$ *is closed.*

(a) *Modify the proof from class to show that if* $Y$ *is Hausdorff and* $X$ *is compact subset of* $Y$*, then* $X$ *is closed.*

(b) *Show that the assumption that* $Y$ *is Hausdorff is necessary.*

*Solution.* □

**Problem 3.** *Let* $X, Y$ *be topological spaces. Prove that if* $f : X \to Y$ *is continuous, then its graph*

$$G_f = \{(x, y) : y = f(x)\}$$

*is a closed subset of* $X \times Y$. [1]

*Solution.* □

**Problem 4.** *Prove that* $S^2$ *is not topologically equivalent to* $\mathbb{R}^2$.

*Solution.* □

**Problem 5.** *Let* $X$ *be a metric space. For subsets* $A, B \subset X$ *define* $d(A, B)$ *as the infimum of* $d(a, b)$ *where* $a \in A$ *and* $b \in B$.

(a) *Find two disjoint closed subsets of* $\mathbb{R}^2$ *so that* $d(A, B) = 0$.

(b) *Show that if* $A, B$ *are disjoint, closed, and* $B$ *is compact, then* $d(A, B) > 0$. [2]

*Solution.* □

**Problem 6.** *Let* $GL_2(\mathbb{R})$ *denote the group of invertible linear maps* $\mathbb{R}^2 \to \mathbb{R}^2$*, topologized as a subset of* $2 \times 2$ *matrices* $M_2(\mathbb{R}) \cong \mathbb{R}^4$.

(a) *Show that matrix multiplication* $M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R})$ *and matrix inversion* $GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ *are continuous.*[3]

(b) *Show that* $GL_2(\mathbb{R}) \subset \mathbb{R}^4$ *is an open subset.* [4]

(c) *Let* $O(n)$ *denote the group of linear isometries of* $\mathbb{R}^n$*, topologized as a subset of* $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. *Prove that* $O(n)$ *is compact.*

*Solution.* □

[1] The converse of this is also true if $Y$ is compact, Hausdorff. This is known as the Closed Graph Theorem.

[2] Hint: construct an open cover of $B$ that is disjoint from $A$. It may help to "shrink" the open sets in the cover before passing to a finite subcover.

[3] Remark: This exercise shows $GL_2(\mathbb{R})$ is a "topological group", which is defined as a group with a topology for which multiplication and inversion are continuous.

[4] Hint: the easiest route involves continuity of a certain polynomial.