Homework 2

Math 141

Due September 25, 2020 by 5pm

Topics covered: topological spaces, topological equivalences, metric spaces, surfaces

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *Let $X_s = \mathbb{R}$ with the standard topology, let $X_d = \mathbb{R}$ with the discrete topology, let $X_i = \mathbb{R}$ with the indiscrete topology, and let $X_c = \mathbb{R}$ with the cofinite topology. For each of these topologies, determine when the identity map $\mathbb{R} \to \mathbb{R}$ is continuous (of the 16 possibilities).*

Solution. □

Problem 2. *Let X be a metric space, and let $C(X)$ denote the set of continuous real-valued functions on X. We say X is "nice" if every such function $X \to \mathbb{R}$ is bounded and achieves a maximum value.*[1] *This allows us to put a metric/topology on $C(X)$ as in class. Fix nice metric spaces X, Y, and let $\phi : X \to Y$ be any continuous map. Define*

$$\phi^* : C(Y) \to C(X)$$

by $\phi^*(f) = f \circ \phi$. Prove that ϕ^* is continuous.

Solution. □

Problem 3. *Let X be the surface (with boundary) obtained by gluing two cylinders, as in discussion session.*

(a) Observe that the boundary of X is a circle.[2] *If one glues a disk to X along its boundary, one obtains a surface (without boundary). How does this resulting surface fit into the classification of surfaces?*

(b) Repeat for the surface with boundary Y obtained by gluing a cylinder to a Möbius band and for the surface Z obtained by gluing two Möbius bands.

Solution. □

Problem 4. *True or false:*

(a) There is a metric on $X = \mathbb{R}$ so that the induced topology is the discrete topology.

(b) There is a metric on $X = \mathbb{R}$ so that the induced topology is the indiscrete topology.

Solution. □

Problem 5 (Armstrong 1.13)**.** *View S^2 as the unit sphere in $\mathbb{R}^3$. Denote $e_3 = (0, 0, 1)$.*

(a) Prove that for each $p \in S^2$ there exists a topological equivalence $f : S^2 \to S^2$ so that $f(p) = e_3$.[3]

(b) Prove that for $p, q \in S^2$ there exists a topological equivalence $f : S^2 \to S^2$ so that $f(p) = q$.

Solution. □

[1] Eventually "nice" will be replaced with a technical condition called *compactness*.

[2] We haven't formally defined the boundary, but hopefully it is intuitively clear. For comparison, the boundary of a cylinder is two circles, and the boundary of the Möbius band is a single circle.

[3] Hint: it maybe helpful to work in spherical coordinates.

Problem 6. *Each of the following configuration spaces is topologically equivalent to a space we've seen before. Identify this space (you should give an explanation for why your answer is correct).*

(a) The configuration space of 2 *unit-length linked rods with one endpoint fixed at the origin and the other endpoint free.*[4]

(b) The configuration space of 4 *unit-length linked rods with endpoints fixed at distance three.*[5]

Suggestion: build your own physical model of this linkage, and play with it. Submit a picture of your model for extra credit.

Solution. □

[4]In this case you should give an explicit topological equivalence.
[5]Hint: Here you may not want to define an explicit topological equivalence.