Homework 1

Math 141

Due September 18, 2020 by 5pm

Topics covered:

Instructions:

- This assignment must be submitted on GradeScope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from Armstrong's book, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1 (Armstrong 1.3)**.** *Let G be a connected graph. Show that there exists a tree $T \subset G$ that contains every vertex of G.*

Solution. □

Problem 2 (Armstrong 1.4-5)**.** *Find a maximal tree in the torus polyhedron from class. Determine the dual graph G' (as in the proof of Euler's theorem) and show that G is not a tree. What is the surface obtained by thickening G'?*

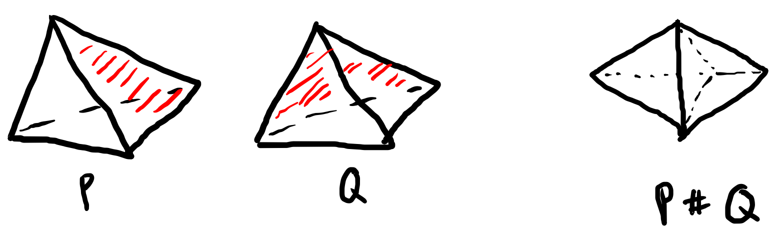
Solution. □

Problem 3 (Armstrong 1.8)**.** *Find a polyhedron that is topologically equivalent to the surface of a pretzel. Compute its Euler number.*

Solution. □

Problem 4.

(a) *Let P, Q be polyhedra. Assume P, Q both contain a n-gon for some n. We form a new polyhedron by removing the (inside of the) n-gon from P and Q and gluing the resulting spaces along the boundary of the n-gon. This process for $P = Q$ both tetrahedra is pictured below. The result is called the connected sum of P and Q, denoted $P\#Q$.*



Prove the formula $\chi(P\#Q) = \chi(P) + \chi(Q) - 2$.

(b) *For each even integer $k \leq 2$, find polyhedron P with $\chi(P) = k$.*

Solution. □

Problem 5 (Armstrong 1.12)**.** (a) *Write a formula for stereographic projection*

$$f : S^1 \setminus \{(0,1)\} \to \mathbb{R}$$

where we view S^1 as the unit circle and $\mathbb{R}$ as the x-axis in $\mathbb{R}^2$.

(b) *Write a formula for f^{-1}.*

Are f and f^{-1} continuous?

Solution. □

Problem 6 (Armstrong 1.6-7)**.** *We say a polyhedron P is regular if there exist p, q so that each face of P has p edges and q faces meet at each vertex. For example, the cube is a regular polygon with $p = 4$ and $q = 3$.*

(a) Let P be a regular polyhedron. Show that if $\chi(P) = 2$, then one has the following formula.

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

(b) Deduce that there are only five regular polyhedra with $\chi(P) = 2$. [1] *What are they?* [2]

Solution. □

[1]Hint: Observe that p, q can't be very large if the equality is to hold. Make this precise.

[2]Hint: once you find p, q that satisfy the equality, you want to find polyhedra that realize these values. If you get stuck, look up "platonic solids".