Math 25b: Theoretical Real Analysis Lecture Notes[1]

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Contents

[1] It's very possible there are errors here. If you find any, please email bmyers@college.harvard.edu

1/28/2019 - Continuity, the real numbers, least upper bound property

The least upper bound property

Here are two facts that are true for $F = \mathbb{R}$ but false for $F = \mathbb{Q}$:

1. Existence of square roots: for $a \in F$, $a > 0$, there exists some $b \in F$ such that $b^2 = a$
2. The intermediate value theorem: let $f : [a, b] \cap F \to \mathbb{R}$ (where $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$) be a continuous function. Assume $f(a) < 0$ and $f(b) > 0$. Then there exists some $c \in (a, b) \cap F$ so that $f(c) = 0$.

Let's first examine the existence of square roots. In 25a, we proved that there doesn't exist a rational $b \in \mathbb{Q}$ such that $b^2 = 2$. We also stated the fundamental theorem of algebra,[2] which says that the polynomial $x^2 - 2$ has a root (so there does exist some number whose square is 2). We discussed the Babylonian method for approximating $\sqrt{2}$ as well (see your notes from 25a for details). There is a less direct approach of proving the existence of $\sqrt{2}$ using the IVT (intermediate value theorem):

Proof. Consider the function

$$\begin{aligned} f : [1, 2] &\to \mathbb{R} \\ x &\mapsto x^2 - 2 \end{aligned}$$

Observe that $f(1) = -1 < 0$ and $f(2) = 2 > 0$. So the IVT says that there exists $c \in (1, 2)$ with $f(c) = 0$, which means $c^2 - 2 = 0$, so c is the square root of 2. $\square$

This proof also gives a counterexample that shows $\mathbb{Q}$ doesn't satisfy the IVT, since there is no rational c with $c^2 = 2$.

Definition. *A subset $A \subset \mathbb{R}$ is **bounded above** if there exists $z \in \mathbb{R}$ such that $a < z$ for all $a \in A$. We call z an **upper bound** for A. Furthermore, we say z is the **least upper bound** of A if $z \leq z'$ for any other upper bound z'. In this case, we write $z = \sup A$.*

Examples

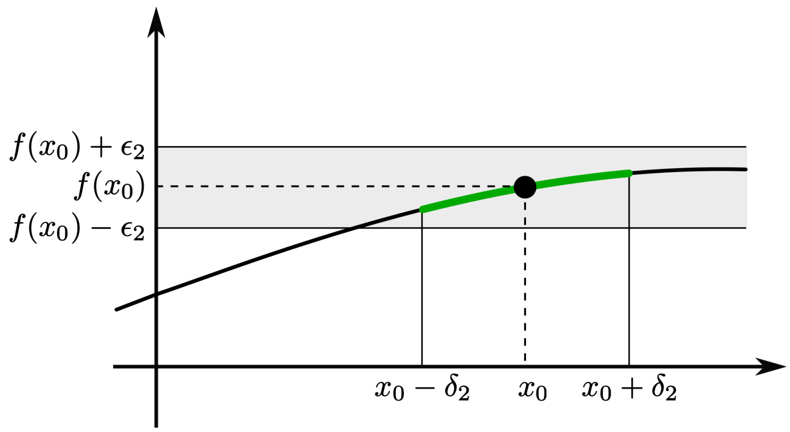
- If $A = \{1, 2, 3\}$, then $\sup A = 3 = \max A$.
- If $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, then $\sup A = 1$ (note that this set doesn't have a maximum).
- If $A = \{x \in \mathbb{Q} : x^2 < 2\}$, then $\sup A = \sqrt{2}$.
- If $B = \mathbb{N}$, then $\sup B$ does not exist (B has no upper bound).
- If $C = \emptyset$, then $\sup C$ does not exist (every $z \in \mathbb{R}$ is an upper bound).

Theorem. ***(least upper bound property)*** *If $A \subset \mathbb{R}$ is nonempty and bounded above, then A has a least upper bound ($\sup A$ exists).*

Remark. *$\mathbb{R}$ is the unique ordered field $(F, +, \cdot, <)$ with the least upper bound property.*

[2] We'll prove this in 25b!

Definition. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is* ***continuous at a*** *if for every $\epsilon > 0$ there exists some $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - f(x)| < \epsilon$.*



f is not continuous at a if there exists some $\epsilon > 0$ such that for every $\delta > 0$ there exists x with $|x - a| < \delta$ and $|f(x) - f(a)| > \epsilon$.

Definition. *A function f is* ***continuous*** *if it is continuous at each point a in the domain.*

Most common functions are continuous. For example, polynomials, trigonometric functions, exponentials, absolute values, etc. are all continuous.

We will now prove the IVT, but we first need a lemma.

Lemma. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at a. If $f(a) > 0$, then f is positive near a. Precisely, this means there exists $\delta > 0$ so that $|x - a| < \delta$ implies $f(x) > 0$.*

Proof. To apply continuity, we want to choose a specific ϵ. Choose $\epsilon = \frac{f(a)}{2}$. By the continuity of f there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \frac{f(a)}{2}$. In particular, this means

$$-\frac{f(a)}{2} < f(x) - f(a) < \frac{f(a)}{2}$$

$$0 < \frac{1}{2}f(a) < f(x) < \frac{3}{2}f(a)$$

So $|x - a| < \delta$ implies $f(x) > \frac{1}{2}f(a) > 0$, as desired. □

Proof. (of the IVT) We have a continuous $f : [a, b] \to \mathbb{R}$ and we know that $f(a) < 0 < f(b)$. We want some $c \in (a, b)$ with $f(c) = 0$. The idea will be to find the 'last' point where f is negative. Consider

$$A = \{x \in [a, b] : f(x) < 0\}$$

A is nonempty, since $f(a) < 0$, and A is bounded above by b. Then the least upper bound property implies that $c = \sup A$ exists.

The claim is that $f(c) = 0$. Either $f(c) = 0, f(c) < 0$, or $f(c) > 0$. We will rule out the latter two possibilities.

First suppose $f(c) > 0$. Then by the lemma, there exists $c' < c$ which is also an upper bound for A (for example, we could choose $c' = c - \frac{\delta}{2}$ where δ is as in the statement of the lemma). This contradicts the fact that c is the *least* upper bound for A.

Now suppose $f(c) < 0$. Then again by the lemma, we know f is negative on some open interval around c, which means that there exists some $a > c$ with $f(a) < 0$. This contradicts the fact that c is an upper bound for A. Therefore $f(c) = 0$. $\square$

The IVT is a foundational part of calculus, and part of this course will be spent understanding in what ways it can be generalized.

Dedekind cuts

Question: What is $\mathbb{R}$?

We could try define $\mathbb{R}$ as the set of decimal expansions, but this has problems. There are multiple ways of representing the same number, and it's difficult to do arithmetic and understand elements of $\mathbb{R}$ this way.

Idea: We understand $\mathbb{Q}$ well, so we will build $\mathbb{R}$ from $\mathbb{Q}$ by 'filling in the gaps' missing in the rational numbers (for example, the square roots).

Definition. *A* ***cut*** *is a subset $\alpha \subset \mathbb{Q}$ that is*

- *nonempty and proper*
- *leftward-closed, in that if $a \in \alpha$ and $b < a$, then $b \in \alpha$*
- *α has no largest element, which means that for each $a \in \alpha$ there exists some $b \in \alpha$ with $b > a$*

Examples

- $\alpha = \{x \in \mathbb{Q} : x < \frac{1}{2}\}$ is a cut.
- $\{x \in \mathbb{Q} : x^2 < 2\}$ is not a cut, since it is not leftward-closed (as it contains 0 but no rationals less than $-\sqrt{2}$). We can make it a cut by instead defining

$$\alpha = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$$

Next time we will define $\mathbb{R}$ as the set $\{\text{cuts } \alpha \subset \mathbb{Q}\}$ and make this set an ordered field, along with showing that $\mathbb{R}$ has the least upper bound property.

1/30/2019 - Skeleton of calculus, continuity theorems, Dedekind cuts

Continuity theorems

Recall that a function $f : [a, b] \to \mathbb{R}$ is continuous if for every point $c \in [a, b]$, f is continuous at c. This means that for any $\epsilon > 0$, there exists some $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

Examples

Take $f(x) = x^2$. Let's show that f is continuous at $c = 2$. Fix $\epsilon > 0$. We want to find $\delta > 0$ such that $|x-2| < \delta$ implies $|x^2-4| < \epsilon$. In general, we want to 'rewrite' the second expression to obtain the first. We can rewrite

$$|x^2 - 4| = |x - 2| \cdot |x + 2|$$

If $|x - 2| < 1$, then $|x + 2| < 5$ (since x must be between 1 and 3). Therefore

$$|x^2 - 4| = |x - 2| \cdot |x + 2| < 5 \cdot |x - 2|$$

So if $|x - 2| < \frac{\epsilon}{5}$, then $5|x - 2| < \epsilon$. Then for $\delta = \min\{1, \frac{\epsilon}{5}\}$, we have that $|x - 2| < \delta$ implies $|x^2 - 4| < \epsilon$.

Let's now show that f is continuous at $c = 20$. Fix $\epsilon > 0$. We want to find $\delta > 0$ such that $|x - 20| < \delta$ implies $|f(x) - 400| < \epsilon$. We have

$$|x^2 - 400| = |x - 20| \cdot |x + 20| < 41 \cdot |x - 20|$$

If we take $\delta = \min\{1, \frac{\epsilon}{41}\}$, then $|x - 20| < \delta$ implies $|x^2 - 400| < \epsilon$.

Note that the δ we found for $c = 20$ is smaller than for $c = 2$, as the graph of $f(x)$ is 'becoming steeper' at $x = 20$ then at $x = 2$, so we must consider a smaller region in the domain.

Note that δ should not depend on x (although it can depend on c).

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then we have the following three theorems:*

1. *(IVT) If $f(a) < d < f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = d$.*
2. *(Boundedness theorem) There exists constants $m \le M$ such that for all $x \in [a, b]$, we have $m \le f(x) \le M$.*
3. *(Max/min value theorem) There exists points $c_1, c_2 \in [a, b]$ so that for every $x \in [a, b]$, we have $f(c_1) \le f(x) \le f(c_2)$.*

In some sense, these are the three most important theorems of the course. What we do from now on will build on them, and we will see to what extent they can be generalized.

Sample application of the IVT

Theorem. *Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in Poly(\mathbb{R})$. If n is odd, then p has a real root, namely there exists some $c \in R$ such that $p(c) = 0$.*

Proof. Factor p for

$$p(x) = x^n(1 + \frac{a_{n-1}}{x} + \ldots + \frac{a_0}{x^n})$$

When x is large, the terms in the parentheses will be very small. If $|x| >> 0$, then $p(x)\ x^n$. Hence there exists some $a, b \in \mathbb{R}$ such that $p(a) < 0 < p(b)$. By the IVT, there exists some $c \in (a, b)$ with $p(c) = 0$. $\square$

Continuity is essential for each of these theorems. For example, consider the function

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ -1 & x = 0 \end{cases}$$

This function doesn't take intermediate values in $(-1, 0)$.

Note also that the boundedness theorem (and min/max value theorem) is false if we replace $[a, b]$ with (a, b). The same function f from above demonstrates this with the interval $(0, 1)$.

The min/max value theorem is a stronger statement than the boundedness theorem (for example, the function $g(x) = x^2$ on $(0, 1)$ satisfies the boundedness theorem, as $0 \leq g(x) \leq 1$, but not the min/max value theorem). We'll now prove these, but we first need a lemma.

Lemma. *Let $f : [a, b] \to \mathbb{R}$ be a function. If f is a continuous at a point $c \in [a, b]$, then f is bounded near c. Precisely, there exists $\delta > 0$ and a constant M such that $|x - c| < \delta$ implies $|f(x)| < M$.*

Proof. Since f is continuous at c, for $\epsilon = 1$ there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < 1$. Rewriting this yields

$$-1 < f(x) - f(c) < 1$$

$$f(c) - 1 < f(x) < f(c) + 1$$

Take $M = \max\{|f(c) - 1|, |f(c) + 1|\}$. Then whenever $|x - c| < \delta$ we have $|f(x)| < M$. $\square$

Recall the least upper bound property for R, which says that if $A \subset \mathbb{R}$ is nonempty and bounded above then A has a least upper bound ($\sup A$ exists).

Proof. (of the boundedness theorem) We will show there is a constant M such that $f(x) \leq M$ for all $x \in [a, b]$. To do this, we will try to find the 'last' point z where f is bounded on the interval $[a, z]$. Consider the set

$$A = \{x \in [a, b] : f \text{ is bounded above on the interval } [a, x]\}$$

A is nonempty, since f is certainly bounded on the interval $[a, a]$ (which is just a single point). A is also bounded above by b Then by the least upper bound property, we can take $z = \sup A$.

We want to show $z = b$. Suppose, for contradiction, that $z < b$. By the lemma, there exists some $\delta > 0$ such that f is bounded on $(z - \delta, z + \delta)$. Let this bound be M_1.

We also know that there exists some point $y \in A \cap (z - \delta, z]$ (if not, then $z - \delta$ is an upper bound for A that is strictly smaller than z, which contradicts the fact that z is the *least* upper bound of A). So f is bounded on the interval $[a, y]$, by definition of the fact that $y \in A$. Let this bound be M_2.

We can just take the maximum of M_1 and M_2 so that f is bounded on the entire interval $[a, z + \delta)$, which means there is a point in A strictly greater than z. This contradicts the fact that z is an upper bound for A. Hence $z = b$, so b is the least upper bound of A.

Final part of the proof (not presented in class)

It is still necessary to demonstrate that $b \in A$ (namely that f is bounded on $[a, b]$ inclusive). By the lemma, f is bounded near b, which means there exists some $\delta > 0$ such that f is bounded by a constant M_1 on $(b - \delta, b]$.

Since b is the *least* upper bound of A, there is some point $y \in A \cap (b - \delta, b]$, which means that f is bounded by a constant M_2 on $[a, y]$ by definition of A.

We can just take the maximum of M_1 and M_2 so that f is bounded on the entire interval $[a, b]$, which completes the proof (and shows that $b \in A$ as desired).

□

The max/min value theorem is proved similarly. We know $Y = \{f(x) : x \in [a, b]\}$ is nonempty and bounded above by the boundedness theorem. Take $M = \sup Y$, and consider

$$A = \{x \in [a, b] : f(x') < M \text{ for all } x' \in [a, x]\}$$

Then show that $f(c) = M$ for $c = \sup A$. (This is a good exercise to try. Hint: use the continuity of f at c.)

Least upper bound property

Lemma. $\mathbb{N} \subset \mathbb{R}$ *is not bounded above*

You might wonder, why do we have to prove something like this? To motivate the discussion, consider

$$\mathrm{Rat}(\mathbb{R}) = \left\{ \frac{p(x)}{q(x)} : p, q \in \mathrm{Poly}(\mathbb{R}), q \neq 0 \right\}$$

This is a field (we can appropriately define addition and multiplication). For $f, g \in \mathrm{Rat}(\mathbb{R})$, we say $f < g$ if there exists some $\delta > 0$ such that $f(x) < g(x)$ for all $x \in (0, \delta)$.

We have a copy of $\mathbb{N}$ contained in $\mathrm{Rat}(\mathbb{R})$ given by the nonnegative constant functions. However, $\mathbb{N} \subset \mathrm{Rat}(\mathbb{R})$ is in fact bounded above by the rational function $\frac{1}{x}$, as we can always find a small $\delta > 0$ for which $\frac{1}{x}$ is greater than a natural number n on the interval $(0, \delta)$.

Now we turn to proving the lemma.

Proof. Suppose, for contradiction, that $\mathbb{N} \subset \mathbb{R}$ is bounded above. Then by the least upper bound property, there is some $z = \sup \mathbb{N}$. So $n \leq z$ for all $n \in \mathbb{N}$. But then $n + 1 \leq z$ for all $n \in \mathbb{N}$, which means $n \leq z - 1$ for all $n \in \mathbb{N}$. Hence $z - 1$ is an upper bound of $\mathbb{N}$, which contradicts the fact that z is the *least* upper bound of $\mathbb{N}$. $\square$

To prove that $\mathbb{R}$ has the least upper bound property, we will give a construction of $\mathbb{R}$. Last time we defined a cut as a subset $\alpha \subset \mathbb{Q}$ that is

- nonempty and proper
- leftward-closed
- has no largest element

We will define

$$\mathbb{R} = \{\text{cuts } \alpha \subset \mathbb{Q}\}$$

Note that $\mathbb{Q} \subset \mathbb{R}$, so for any $r \in \mathbb{Q}$ there is a cut $r^* = \{x \in \mathbb{Q} : x < r\}$.

Theorem. *Defined this way, $\mathbb{R}$ is an ordered field with the least upper bound property.*

Recall that an ordered field is a quadruple $(F, +, \cdot, <)$. The addition and multiplication make F into a field, and $<$ must satisfy a few axioms:

- $<$ is transitive, in that $a < b$ and $b < c$ implies $a < c$
- $<$ is a total order, in that either $a < b$, $b < a$, or $a = b$
- $<$ is compatible with the field operationss, in that $a < b$ implies $a + c < b + c$ and, when $c > 0$, we have $a \cdot c < b \cdot c$

Next time, we will define the addition, multiplication, and order on this set, along with justifying that these definitions satisfy the appropriate axioms. We will also prove the least upper bound property.

2/4/2019 - Convergence, sequences, limits, continuity

Construction of $\mathbb{R}$

Recall that we defined

$$\mathbb{R} = \{\text{cuts } \alpha \subset \mathbb{Q}\}$$

where a cut is a proper, nonempty subset of $\mathbb{Q}$ that is leftward-closed and contains no largest element. Note that there is a natural injection $i : \mathbb{Q} \hookrightarrow \mathbb{R}$ given by associating $r \in \mathbb{Q}$ with the cut $i(r) = r^* = \{x \in \mathbb{Q} : x < r\}$.

Theorem. *$\mathbb{R}$ is an ordered field with the least upper bound property.*

Proof. First we will define an order $<$ on the set $\mathbb{R}$. For two cuts $\alpha, \beta \in \mathbb{R}$, we will say $\alpha < \beta$ if $\alpha \subsetneq \beta$ as subsets of $\mathbb{Q}$. The order $<$ should be transitive. This is clear.

$<$ should also be a total order, which means that for any $\alpha, \beta \in \mathbb{R}$ either $\alpha < \beta$, $\beta < \alpha$ or $\alpha = \beta$. To see that this holds, suppose for contradiction that there exists $\alpha, \beta \in \mathbb{R}$ such that none of these are true. Since $\alpha \not< \beta$, there exists $b \in \beta \cap \alpha^c$. Similarly, there exists $a \in \alpha \cap \beta^c$. a, b are rational and $\mathbb{Q}$ is an ordered field, so either $a < b$, $b < a$, or $a = b$. We certainly can't have equality by choice of a and b. And $a < b, b < a$ are impossible because cuts are leftward closed (for example, if $a < b$, since $b \in \beta$ we would also have $a \in \beta$, which is a contradiction).

We can now prove the least upper bound property. Let $A \subset \mathbb{R}$ be nonempty and bounded above (namely, there is some cut $\beta \in \mathbb{R}$ such that $\alpha < \beta$ for all $\alpha \in A$). Consider

$$\gamma = \bigcup_{\alpha \in A} \alpha \subset \mathbb{Q}$$

First observe that γ is indeed a cut. It is nonempty, proper because A is bounded by β, leftward-closed because each α is leftward-closed, and has no maximum element since none of the cuts α do.

γ is the least upper bound for A. Since $\alpha \subset \gamma$, we have $\alpha \leq \gamma$ for all $\alpha \in A$. If γ' is also an upper bound for A, then $\alpha \subset \gamma'$ for all $\alpha \in A$. This implies that $\bigcup \alpha \subset \gamma'$, so $\gamma \subset \gamma'$. Hence $\gamma \leq \gamma'$. Therefore γ is the least upper bound of A.

Next we want to make $\mathbb{R}$ into a field. Define

- $\alpha + \beta = \{x \in \mathbb{Q} : x < a + b \text{ for some } a \in \alpha, b \in \beta\}$
- If $\alpha, \beta > 0^*$, then define $\alpha \cdot \beta = \{x \in \mathbb{Q} : x < a \cdot b \text{ for some } a \in \alpha, b \in \beta \text{ with } a, b > 0\}$
- If $\alpha < 0^*$ and $\beta > 0^*$, then define $\alpha \cdot \beta = -((-\alpha) \cdot \beta)$.
- Note that $-\alpha$ is the unique cut with the property $\alpha + (-\alpha) = 0^*$. It is given by $-\alpha = \{x \in \mathbb{Q} : -x \text{ is less than all } a \in \alpha, \text{ but } x \text{ is not the largest such value}\}$

These operations make $\mathbb{R}$ into an ordered field. Moreover, they restrict to the usual operations on $\mathbb{Q} \subset \mathbb{R}$. This is easy to check and will be left as an exercise. $\square$

Application of this construction to the existence of $\sqrt{2}$

We will show that $\sqrt{2}$ does in fact exist. Consider

$$\alpha = \{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$$

α is a cut, and we claim that $\alpha^2 = 2^*$. First observe that $\alpha^2 \leq 2^*$. This is because

$$\alpha^2 = \{x \in \mathbb{Q} : x < ab : a, b \in \alpha \text{ and } a, b > 0\}$$

We want to show $\alpha^2 \subset 2^*$. Fix $x \in \alpha^2$. Then $x < ab$ for $a, b \in \alpha$ and $a, b > 0$. Without loss of generality assume $a \leq b$. Then

$$x < ab < b^2 < 2$$

so $x \in 2^*$, which shows that $\alpha \leq 2^*$.

Now suppose for a contradiction that $\alpha^2 < 2^*$. Then by the continuity of the map $x \mapsto x^2$ on $\mathbb{R}$, there exists some $r \in \mathbb{Q}$ so that $r^* > \alpha$ and $\alpha^2 < (r^*)^2 < 2$.[a] However, this means $r \in \alpha$ by definition of α, so $r^* < \alpha$, which is a contradiction.

[a]This follows because the continuous function $x \mapsto x^2 - 2^*$ is negative at α by assumption. Then it is negative around α, so we can choose a point r^* to the right of α for which the function is still negative.

Sequences and continuity

Sequences will give us an easier way to show that a function is not continuous.

Definition. *A **sequence** in $\mathbb{R}^n$ is an ordered list $(a_1, a_2, \ldots)$ with each $a_i \in \mathbb{R}^n$.*

Examples

- $(1, 2, 4, 8, 16, \ldots)$
- $(1, 1.4, 1.41, 1.414, \ldots)$
- $(1, -1, 1, -1, \ldots)$
- $a_n = (\frac{1}{n}, n) \in \mathbb{R}^2$

Definition. *A sequence (a_n) **converges** to $p \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exist some N such that $n \geq N$ implies $|a_n - p| < \epsilon$. If (a_n) converges to p, then we write $a_n \to p$.*

Note that for points $x, y \in \mathbb{R}^n$, the notion of the distance $|x - y|$ in $\mathbb{R}^n$ is given by

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

Definition. *Given an increasing sequence $1 \leq n_1 < n_2 < \ldots$, we define the **subsequence** $(a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$ of $(a_1, a_2, a_3, \ldots)$.*

Examples

- A subsequence of $(1, 2, 4, 8, \ldots)$ is $(1, 4, 16, 64, \ldots)$
- The sequence $(1, 2, 4, 8, \ldots)$ does not converge.
- The sequence $(1, 1.4, 1.41, \ldots)$ converges to $\sqrt{2}$.
- The sequence $(1, -1, 1, -1, \ldots)$ does not converge, but the subsequence (a_{2k}) does converge.

If a_n converges to p, then a_{n_k} converges to p. However, it is possible that a_{n_k} converges to p even though a_n does not converge to p.

Sequences give us a way to think about continuity:

Theorem. *The following are equivalent:*

1. *f is continuous at p.*
2. *For every sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}^n$ that converges to p, the sequence $(f(a_1), f(a_2), \ldots)$ converges to $f(p)$.*

So if we want to approximate the value of a continuous function at p, we can look at the values of f on points near p.

Examples

- Consider the function
$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$
f is not continuous at 0. To show this using the theorem, consider the sequence given by $a_n = \frac{1}{n}$. (a_n) converges to 0, but $(f(a_n))$ does not converge to $f(0) = 1$, since $f(a_n) = 0$ for all a_n.
- Consider the function
$$g(x) = \begin{cases} \sin(\frac{1}{x}) & x > 0 \\ 0 & x = 0 \end{cases}$$
Note that
$$g(\frac{2}{n\pi}) = \begin{cases} 0 & n \text{even} \\ 1 & n = 4k + 1 \\ -1 & n = 4k + 3 \end{cases}$$
(This should give a rough idea what the graph of g looks like.) g is not continuous at 0. Consider the sequence given by $a_n = \frac{2}{n\pi}$. (a_n) converges to 0, but $(f(a_n))$ is the sequence $(1, 0, -1, 0, 1, 0, \ldots)$, and it does not converge (to 0).

Now we will prove the theorem.

Proof. Suppose f is continuous at p. Let (a_n) be a sequence converging to p. Fix $\epsilon > 0$. We want to show that there exists $N > 0$ such that $n \geq N$ implies $|f(a_n) - f(p)| < \epsilon$. Since f is continuous at p, there exists some $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - f(p)| < \epsilon$. We know that the sequence (a_n) converges to p, so there exists some $N > 0$ such that $n \geq N$ implies $|a_n - p| < \delta$. For $n \geq N$, we have $|a_n - p| < \delta$, which implies $|f(a_n) - f(p)| < \epsilon$.

Now we will prove the other direction by contrapositive. Suppose that f is not continuous at p. We will construct a sequence (a_n) which converges to p, but whose image $(f(a_n))$ does not converge to $f(p)$. Since f is not continuous at p, there exists $\epsilon > 0$ so that for all $\delta > 0$ there exists x such that $|x - p| < \delta$ and $|f(x) - f(p)| > \epsilon$. Then for $\delta = \frac{1}{n}$, choose such a sequence of points (a_n) so that $|a_n - p| < \frac{1}{n}$ and $|f(a_n) - f(p)| > \epsilon$. The sequence (a_n) converges to p, but the images $(f(a_n))$ do not converge to $f(p)$ (since they are always at least ϵ away). $\square$

Let's try the following:

Theorem. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous at p and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(p)$, then $g \circ f : \mathbb{R} \to \mathbb{R}$ is continuous at p.*

Proof. Given a sequence (a_n) that converges to p, we want to show that $((g \circ f)(a_n))$ converges to $(g \circ f)(p)$. Since f is continuous at p, $f(a_n)$ converges to $f(p)$. Since g is continuous at $f(p)$, $g \circ f(a_n)$ converges to $g \circ f(p)$, as desired. $\square$

2/6/2019 - Topology, open/closed sets, compactness/Heine-Borel, continuity

Real numbers and decimals

Fact. $0.999\overline{9} = 1.000\overline{0}$. What does this mean in our language? Define the cut

$$x = \bigcup_{k \geq 1} (\frac{9}{10} + \ldots + \frac{9}{10^k})^* = \bigcup_{k \geq 1} (1 - \frac{1}{10^k})^* \subset \mathbb{Q}$$

Lemma. $x = 1^*$

Proof. We will show both inequalities. The direction $x \leq 1^*$ is easier, since every $r \in x$ is less than 1 by definition.

It remains to show $1^* \leq x$, so for each $r \in \mathbb{Q}$ with $r < 1$ there exists some k so that $r < 1 - \frac{1}{10^k}$ (this is what $r \in x$ means). Write $r = \frac{a}{b}$ in lowest terms, with $a \in \mathbb{Z}, b \in \mathbb{N}$. We know $a < b$, since $r < 1$. Furthermore, we know $b - a \geq 1$ (since a and b are integers, their difference must be at least 1).

So we want to find k such that

$$\frac{a}{b} < 1 - \frac{1}{10^k}$$

$$\frac{1}{10^k} < 1 - \frac{a}{b} = \frac{b-a}{b}$$

Observe that it suffices to show

$$\frac{b-a}{b} \geq \underbrace{\frac{1}{b} \geq \frac{1}{10^b}}_{(*)}$$

You can prove $(*)$ easily by showing that $n < 10^n$ by induction. We can take $k = b + 1$, which proves the lemma. □

We'll turn to another lemma that came up recently.

Lemma. *For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\frac{1}{10^k} < \epsilon$.*

Proof. Fix $\epsilon > 0$. We can rewrite this as $10^k > \frac{1}{\epsilon}$. There exists a $k \in \mathbb{N}$ so that $k > \frac{1}{\epsilon}$. This is because $\mathbb{N} \subset \mathbb{R}$ is not bounded above (we showed this previously). By the claim from above, $10^k > k > \frac{1}{\epsilon}$. □

This lemma also shows that the sequence $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ converges to zero.

Limits and continuity

Even if f is not continuous at a, we can still try to describe behavior near a.

Definition. *We say* $f : \mathbb{R}^n \to \mathbb{R}^m$ ***approaches limit*** L ***near*** a *if for all* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that* $0 < |x - a| < \delta$ *implies* $|f(x) - L| < \epsilon$*. In such a case, we write*

$$\lim_{x \to a} f(x) = L$$

This looks a lot like the definition of continuity. However, note that the value of $f(a)$ is irrelevant when taking the limit of f at a. This is because we demand $0 < |x - a|$, so we are never evaluating f at the point a.

Examples

- Consider the function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = 0$$

By definition, f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.

Theorem. *(Algebra of limits) Fix* $f, g : \mathbb{R}^n \to \mathbb{R}^m$*. Assume* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = K$*. Then we have*

1. $\lim_{x\to a} f(x) + g(x) = L + K$
2. $\lim_{x\to a} f(x)g(x) = LK$ *when* $m = 1$
3. $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}$ *when* $m = 1$ *and* $L \neq 0$

Sample application of the algebra of limits theorem

- The function $x \mapsto x^n$ is continuous because $x \mapsto x$ is continuous (this claim is easier), as x^n is the product of x n times.
- The function $x \mapsto \frac{x^{14} - 3x^{100}}{1 - x^3}$ is continuous everywhere it is defined (this is much easier than showing continuity with ϵ and δ).

Proof. We must use the $\epsilon - \delta$ definition. The three steps of such a proof are (a) algebra, (b) estimation, and (c) write-up.

We will first prove statement 1 of the theorem. Fix $\epsilon > 0$. Use algebra to rewrite

$$\begin{aligned} \big|f(x) + g(x) - (L + K)\big| &= \big|f(x) - L + g(x) - K\big| \\ &\leq \big|f(x) - L\big| + \big|g(x) - K\big| \end{aligned}$$

by the triangle inequality. Now estimate to conclude that we should make each of the two parts less than $\frac{\epsilon}{2}$. By assumption, there exists $\delta_f, \delta_g > 0$ such that $0 < |x - a| < \delta_f$ implies $|f(x) - L| < \frac{\epsilon}{2}$

and $0 < |x-a| < \delta_g$ implies $|f(x) - K| < \frac{\epsilon}{2}$.

Now to write up the solution, take $\delta = \min\{\delta_f, \delta_g\}$. If $0 < |x-a| < \delta$, then

$$\begin{aligned} \big|f(x) + g(x) - (L+K)\big| &\leq \big|f(x) - L\big| + \big|g(x) - K\big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

as desired.

Now we will prove the second statement of the theorem. Write

$$\begin{aligned} \big|f(x)g(x) - LK\big| &= \big|f(x)g(x) - Lg(x) + Lg(x) - LK\big| \\ &\leq \underbrace{\big|f(x) - L\big|}_{(*)} \cdot |g(x)| + |L| \cdot \underbrace{\big|g(x) - K\big|}_{(*)} \end{aligned}$$

We can make $(*)$ small, but we will have to be a bit careful here, since $|g(x)|$ is not actually constant as x varies. The idea will be to find a bound for this value $|g(x)|$. There exists some $\delta_g > 0$ such that $0 < |x-a| < \delta_g$ implies $|g(x) - K| < \frac{\epsilon}{2|L|}$. Then for these values of x,

$$K - \frac{\epsilon}{2|L|} < g(x) < K + \frac{\epsilon}{2|L|}$$

$$g(x) < \underbrace{\max\left\{\left|K - \frac{\epsilon}{2|L|}\right|, \left|K + \frac{\epsilon}{2|L|}\right|\right\}}_{M}$$

There also exists $\delta_f > 0$ such that $0 < |x-a| < \delta_f$ implies $|f(x) - L| < \frac{\epsilon}{2M}$. Take $\delta = \min\{\delta_f, \delta_g\}$. The writeup follows similarly. It's a good idea to try proving statement 3 of the theorem yourself. □

Generalized boundedness theorem

Recall that the boundedness theorem says that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then f is bounded. Precisely, this means that there exists a constant M such that $|f(x)| \leq M$ for all $x \in [0,1]$.

Further recall that if we replace $[0,1]$ with $(0,1)$, the analagous statement is false (for example, consider $f(x) = \frac{1}{x}$).

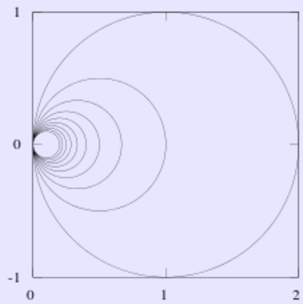
Question. What is the difference between $[0,1]$ and $(0,1)$ that allows the theorem to fail?

Question. For which $X \subset \mathbb{R}^n$ does the boundedness theorem hold?

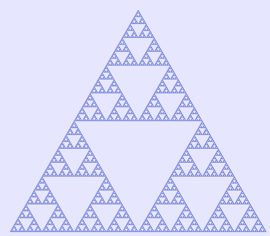
For which of the following does the boundedness theorem hold?

- $X = [0,1]^n$
- $X = \{x \in \mathbb{R}^n : |x| = 1\}$
- $X = \{x \in \mathbb{R}^n : |x| \leq 1\}$

- $X = \{A \in M_n(\mathbb{R}) : A^t A = I\}$ (the isometries/orthogonal matrices)
- The Hawaiian earring:



- The Serpinski triangle:



The tool we used for the boundedness theorem was the least upper bound property. However, this is not enough for higher-dimensional situations. We will need a new tool: compactness.

Topology definitions

Definition. *Let $A \subset \mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is a **limit point** of A if there is a sequence $(a_n) \subset A \setminus \{p\}$ so that (a_n) converges to p.*

Definition. *Let $A \subset \mathbb{R}^n$. If $p \in A$ is not a limit point of A, then p is an **isolated point**.*

Definition. *A subset $A \subset \mathbb{R}^n$ is **closed** if it contains all of its limit points.*

Definition. *A subset $A \subset \mathbb{R}^n$ is **open** if for all $a \in A$, there exists some $r > 0$ such that the ball $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$ is contained in A.*

Examples

- If $A = \{(x, y) : x^2 + y^2 \leq 1\} \cup \{(2, 0)\}$, $(0, 0)$ is a limit point. Take the sequence $(\frac{1}{2}, 0), (\frac{1}{3}, 0), (\frac{1}{4}, 0), \ldots$. However, $(2, 0)$ is not a limit point. Any sequence in $A \setminus \{(2, 0)\}$ cannot converge to $(2, 0)$. So $(2, 0)$ is an isolated point.
- Defined as above, A is closed.
- Note that if we replace $\leq$ with $<$ in the definition of A, A is no longer closed. However, A is not open either (any ball around $(2, 0)$ contains points not in A).
- $\mathbb{R}^n$ and $\varnothing$ are both open and closed.

Note that these notions generalize closed and open intervals in $\mathbb{R}$.

Definition. *A subset $A \subset \mathbb{R}^n$ is* ***sequentially compact*** *if every sequence $(a_n) \subset A$ has a subsequence (a_{n_k}) that converges to some point $p \in A$.*

Examples

- Any finite set is compact. A sequence visits a finite number of points infinitely many times, so there is some $p \in A$ which is visited by the sequence infinitely many times. We can take the subsequence $(p, p, p, \ldots)$.
- $(0, 1) \subset \mathbb{R}$ is not sequentially compact, as the sequence (a_n) given by $a_n = \frac{1}{n}$ converges to $0 \notin (0, 1)$.
- $\mathbb{R} \subset \mathbb{R}$ is not sequentially compact, since the sequence (a_n) given by $a_n = n$ does not converge and has no convergent subsequence.

For the purposes of some intuition, we can view compactness as a generalization of what it means to be finite. For example, functions from finite sets are bounded, as are functions from compact sets.

More generally, note that sequentially compact sets are closed, as we can take a sequence that converges to every limit point of such a set, and by compactness the set must contain that point.

Also observe that sequentially compact sets are bounded. Precisely, this means there exists some $R > 0$ such that $A \subset B_R(0)$ (A is contained in the ball of radius R around 0). If a set isn't bounded, then we can take some sequence of points that are farther and farther from the origin. This sequence has no converging subsequence.

Next time, we will prove that $[0, 1]$ is compact and that the boundedness theorem generalizes to continuous functions on compact sets.

2/11/2019 - More topology, subset trichotomy, compactness and coverings

Sequential compactness

Recall A subset $A \subset \mathbb{R}^n$ is

- *closed* if whenever $(a_n) \subset A$ is a sequence that converges to p, $p \in A$.
- *bounded* if $A \subset B_r(0)$ for some $r > 0$. Equivalently, $A \subset [-t, t]^n$ for some $t > 0$ (since we can fit balls inside boxes and boxes inside balls).
- *sequentially compact* if every $(a_n) \subset A$ has a subsequence $(a_{n_k}) \subset (a_n)$ such that a_{n_k} converges to p with $p \in A$.

We also know that if A is sequentially compact, A is bounded and closed.

Examples

- $\mathbb{R} \times [0, 1] \subset \mathbb{R}^n$ is closed but not bounded. The sequence that goes off to infinitiy has no convergent subsequence.
- $(0, 1)^2$ is bounded but not closed. The sequence that becomes close to the boundary of this set converges to a point outside of the set.

We can use our topological definitions to prove a generalization of the boundedness theorem.

Theorem. *(Generalized boundedness theorem). Let $A \subset \mathbb{R}^d$ be sequentially compact and $f : A \to \mathbb{R}^d$ be a continuous function. Then $f(A) = \{f(a) : a \in A\}$ is sequentially compact. In particular, $f(A)$ is bounded.*

Proof. Fix a sequence $(b_n) \subset f(A)$. Write $b_n = f(a_n)$ for some $a_n \in A$. Since A is sequentially compact, there exists a converging subsequence $(a_{n_k}) \subset (a_n)$ that converges to a point $p \in A$. Since f is continuous, this implies that the subsequence $(b_{n_k}) \subset (b_n)$ converges to the point $f(p) \in f(A)$, by our characterization of continuous functions last class. $\square$

This is a very useful tool, but we will have to develop a way to recognize sequentially compact sets.

Theorem. *A closed interval $[a, b] \subset \mathbb{R}$ is sequentially compact.*

Proof. Fix a sequence $(x_n) \subset [a, b]$. For $c \in [a, b]$, if for all $\epsilon > 0$ there exists n such that $x_n \in (c - \epsilon, c + \epsilon) \setminus \{c\}$, then there exists a subsequence converging to c (choose x_{n_k} to be within $\frac{1}{k}$ of c. Be sure to choose x_{n_k} so that the resulting sequence is indeed an ordered subsequence of (x_n)).

So if there does not exist a subsequence that converges to c, then there exists some $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon)$ contains no points of the sequence. We will now apply the least upper bound property. Define the set

$$A = \{x \in [a,b] : x_n < x \text{ for only finitely many } n\}$$

Certainly $a \in A$ and A is bounded above by b, so by the least upper bound property $c = \sup A$ exists.

Now suppose for contradiction that there is no convergent subsequence to c. Then there exists $\epsilon > 0$ such that no x_n is within $(c-\epsilon, c+\epsilon)$. Then any point in $(c, c+\epsilon)$ is also in A, so this contradicts the fact that c is an upper bound of A. Hence there is a subsequence that converges to c.

Note that we must also consider the case when $c = b$. Then for any ϵ there must be a point in $(c-\epsilon, b)$ □

Theorem. $[0,1]^n \subset \mathbb{R}^n$ *is sequentially compact. In general, closed intervals are compact.*

Proof. The proof is left as an exercise. You can do this by projecting the sequence on to each of the n intervals $[0,1]$ and using the sequential compactness of the closed interval, proceeding by induction. □

Theorem. *(Bolzano-Weierstrass theorem). Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence. Then $(x_n) \subset [-t,t]^n$ for some $t > 0$. The product $[-t,t]^n$ is sequentially compact, so (x_n) has a convergent subsequence. □

Theorem. *(Heine-Borel theorem). A is sequentially compact if and only if A is closed and bounded.*

Proof. We proved the forward direction as a lemma last class, so it remains to show the reverse implication.

Fix $A \subset \mathbb{R}^n$ closed and bounded, and take a sequence $(a_n) \subset A$. A is bounded, so $A \subset [-t,t]^n$ for some $t > 0$. Hence (a_n) has a convergent subsequence (a_{n_k}) that converges to $p \in [-t,t]^n$. However, we also need that $p \in A$. This is true because A is closed. □

Exercise

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = A \cup \{0\}$. Are A, B sequentially compact?

- A is not closed, and hence not sequentially compact. The sequence (a_n) given by $a_n = \frac{1}{n}$ does not converge to a point in A.
- B is sequentially compact. B is closed and bounded. You can also argue directly by finding a convergent subsequence, but this requires some care.

More topology

Recall that a set $U \subset \mathbb{R}^n$ is open if for all $u \in U$, there is some $r > 0$ such that $B_r(u) \subset U$. We will now further explore the relationship between being open and closed.

Lemma. *If $U \subset \mathbb{R}^n$ is open, then U^c is closed. If $A \subset \mathbb{R}^n$ is closed, then A^c is open.*

Proof. Let U be open. Take $(x_n) \subset U^c$ to be a sequence converging to p. We want to show $p \in U^c$. Suppose for contradiction that $p \in U$. U is open, so there exists $r > 0$ such that $B_r(p) \subset U$. However, since (x_n) converges to p, so there is some N such that $n \geq N$ implies that $x_n \in B_r(p)$. This contradicts the assumption that $x_n \in U^c$.

Let A be closed. Suppose A^c is not open. Then there is some $p \in A^c$ such that for all $r > 0$, there is some point $a \in A \cap B_r(p)$. Then there exists a sequence $(a_n) \subset A$ that converges to p (by taking $a_n \in A$ to be some point within $\frac{1}{n}$ of p). This contradicts the assumption that A is closed. $\square$

Remember that subsets of $\mathbb{R}^n$ can be open, closed, both open and closed, or neither open nor closed. However, there is still a trichotomy.

Let $Y \subset \mathbb{R}^n$. Then for $x \in \mathbb{R}^n$, exactly one of the following holds:

- There exists $r > 0$ such that $B_r(x) \subset Y$. In this case we say x is an ***interior point*** of Y.
- There exists $r > 0$ such that $B_r(x) \subset Y^c$. In this case we say x is an ***exterior point*** of Y.
- For every $r > 0$, the ball $B_r(x)$ intersects both Y and Y^c. In this case we say x is a ***boundary point*** of Y.

They we can decompose $\mathbb{R}^n$ into

$$\mathbb{R}^n = \operatorname{int}(Y) \sqcup \operatorname{bd}(Y) \sqcup \operatorname{ext}(Y)$$

Note that Y is open if and only if $Y = \operatorname{int}(Y)$, and Y is closed if and only if $\operatorname{bd}(Y) \subset Y$.

Examples

Let $Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \cup \{(2, 0)\}$.

- The interior of Y is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
- The boundary of Y is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(2, 0)\}$.
- The exterior of Y consists of all the other points.

We can use open and closed sets to characterize continuity.

Lemma. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $U \subset \mathbb{R}^m$ is open implies $f^{-1}(U) \subset \mathbb{R}^n$ is open.*

Covering compactness

Let $\mathscr{U}$ be a collection of open sets in $\mathbb{R}^n$. $\mathscr{U}$ could be finite, countable, or uncountable. Generally we will write

$$\mathscr{U} = \{U_\beta : \beta \in B\}$$

Definition. *$\mathscr{U}$ is an **open cover** of $X \subset \mathbb{R}^n$ if*

$$X \subset \bigcup_{\beta \in B} U_\beta$$

*If $\mathscr{V}, \mathscr{U}$ are both open covers of X with $\mathscr{V} \subset \mathscr{U}$, then $\mathscr{V}$ is a **subcover** of $\mathscr{U}$.*

Definition. *$X \subset \mathbb{R}^n$ is **covering compact** if any open cover of X has a finite subcover.*

Examples

- Let $X = \mathbb{R}$. Then the cover
$$\mathscr{U} = \{(n, n+2) : n \in\}$$
is an open cover of $\mathbb{R}$ with no finite subcover. Hence $\mathbb{R}$ is not convering compact. However, note that $\mathbb{R}$ does have finite covers (for example $\mathscr{U} = \{\mathbb{R}\}$).
- If X is finite, then X is covering compact.

We will prove the following results.

Theorem. *$X \subset \mathbb{R}^n$ is sequentially compact if and only if X is covering compact.*

Theorem. *$[0,1]^n \subset \mathbb{R}^n$, and closed intervals in general, are covering compact.*

Theorem. *(Nested interval theorem). If $Q_k \subset \mathbb{R}^d$ are nested rectangles such that $Q_{k+1} \subset Q$ for all k, then $\bigcap Q_k \neq \varnothing$.*

2/13/2019 - Differentiability, mean value theorem, Taylor polynomials

Compactness

Recall that we have two notions of compactness:

- $X \subset \mathbb{R}^n$ is *sequentially compact* if for all sequences $(x_n) \subset X$, there is a subsequence $(x_{n_k}) \subset (x_n)$ that converges to some point $p \in X$.
- $X \subset \mathbb{R}^n$ is *covering compact* if every open cover of X has a finite subcover. In other words, whenever
$$X \subset \bigcup_{\beta \in B} U_\beta$$
there exists $\beta_1, \ldots, \beta_r \in B$ such that
$$X \subset U_{\beta_1} \cup \ldots \cup U_{\beta_r}$$

We will spend today working on the following theorem:

Theorem. *$[0,1]^n \subset \mathbb{R}^n$ is covering compact.*

Alternate proof of the boundedness theorem

Theorem. *If $f : [0,1] \to \mathbb{R}$ is continuous, then there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$.*

Proof. By continuity, for each $y \in [0,1]$ there exists a δ_y such that $|x - y| < \delta_y$ implies $|f(x) - f(y)| < 1$. Equivalently, this means

$$f(y) - 1 < f(x) < f(y) + 1$$

$$|f(x)| < \max\{|f(y) + 1|, |f(y) - 1|\} = M_y$$

The open sets $\{B_{\delta_y}(y) : y \in [0,1]\}$ give a covering of $[0,1]$. Since $[0,1]$ is covering compact, there exist finitely many $y_1, \ldots, y_r$ such that the balls $B_{y_1}(y_1), \ldots, B_{y_r}(y_r)$ cover $[0,1]$. Then for $x \in [0,1]$ we know that

$$|f(x)| < \max\{M_{y_1}, \ldots, M_{y_r}\} = M$$

which completes the proof. $\square$

As in the above proof, compactness is used as a tool 'to go from the infinite to the finite.' You will prove the following corollary on homework.

Corollary. *X is sequentially compact if and only if X is covering compact.*

We will now turn to prove the theorem, but we will need another fact first, which you will also prove on the homework.

Theorem. *(Nested interval theorem/Onion ring[3] theorem). Let $Q_k \subset \mathbb{R}^n$ be closed rectangles such that $Q_{k+1} \subset Q_k$. Then*

$$\bigcup_k Q_k \neq \varnothing$$

Now we will prove that $[0,1]^n \subset \mathbb{R}^n$ is covering compact.

Proof. Suppose, for contradiction, that there exists a covering $\mathscr{U}$ of $[0,1]^n$ with no finite subcover. Decompose $[0,1]^n$ into 2^n quadrants, based on whether or not a point is in the first or second half of each interval $[0,1]$. One of these quadrants must not have a finite subcover (or else there would be a finite subcover for all of $[0,1]^n$). Let Q_1 be this quadrant.

Inductively define $Q_{k+1} \subset Q_k$ with the properties

1. Q_{k+1} is not covered by finitely many elements of $\mathscr{U}$.
2. The diameter of Q_k is $\frac{\sqrt{n}}{2^k}$ (this is the diagonal of the quadrant).

The onion ring theorem implies that there is some $z \in \bigcap_k Q_k$. Then by definition of the cover $\mathscr{U}$, there is some open $U \in \mathscr{U}$. Since U is open and the diameters of Q_k go to zero, there is some N such that $n \geq N$ implies $Q_n \subset U$. This is a contradiction, since all of these Q_n are contained in the single set U. However, we chose the Q_n to have no finite subcover. Therefore $\mathscr{U}$ does not exist, so $[0,1]^n$ is covering compact. □

This concludes the first part of the course, in which we covered

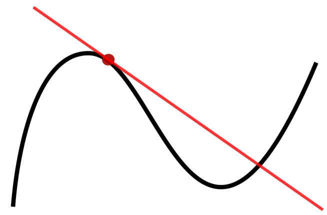
- the continuity theorems (intermediate value theorem, boundedness theorem, min/max value theorem)
- the least upper bound property and compactness
- the compactness theorems (Bolzano-Weierstrass theorem and Heine-Borel theorem)

We will use these foundational results in the next part of the course.

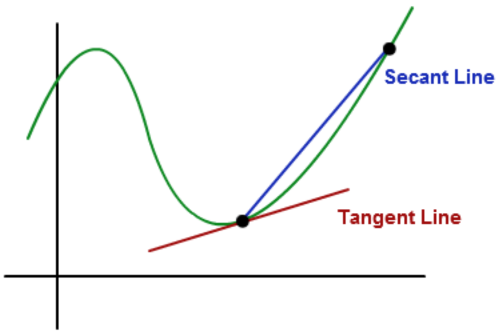
[3]

Differentiability

Question: Given a function $f : \mathbb{R} \to \mathbb{R}$, what is the linear map $\ell(x) = mx + b$ that best approximates f near $a \in \mathbb{R}$?



Assume $a = 0$. Then we want $\ell(0) = f(0)$, so we should take $b = f(0)$. To find m, consider secant lines.



For sufficiently nice f, we would expect that the slope

$$\frac{f(h) - f(0)}{h}$$

converges.

Definition. *A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** at $a \in \mathbb{R}$ if*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Denote this limit by $f'(a)$.

Definition. *A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** if it is differentiable at all $a \in \mathbb{R}$.*

Basic facts about differentiable functions

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions.

- f is continuous.
- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $(f/g)' = \frac{f'g - fg'}{g^2}$
- $(f \circ g)'(a) = f'(g(a))g'(a)$

- If $f(x) = c$ is constant, then $f'(x) = 0$.

These are all not too difficult to prove. We will show the first one now.

Proof. Write

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

using the substitution $x = a + h$. Then we have

$$\begin{aligned}
\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a}\right)(x - a) \\
&= \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a}\right) \cdot \lim_{x\to a} x - a \\
&= f'(a) \cdot 0 = 0
\end{aligned}$$

□

The only tricky one is the chain rule, which we will return to in more generality later.

Corollary. *We can differentiate polynomials and rational functions (quotients of polynomials).*

Examples

- Consider the function

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

  To compute $f'(x)$ away from 0, we can just use the differentiation rules (as the derivative depends on the local behavior of f). But to compute $f'(0)$, we must use the limit definition. By examining the left and right limits we have

$$\lim_{h\to 0+} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$\lim_{h\to 0-} \frac{f(h)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

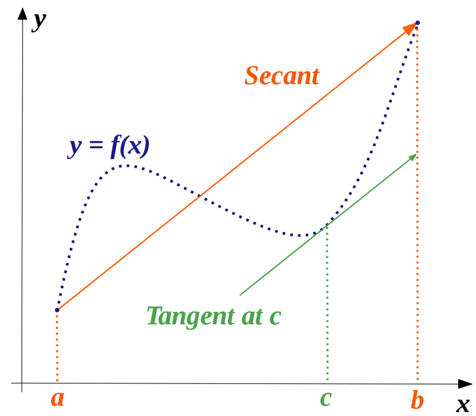
  So the overall limit indeed exists, and we have $f'(0) = 0$.

Differentiation defines a function $D : V \to W$, where V is the set of differentiable functions and W is the set of all functions. In fact, V and W are vector spaces. The rules for sums, products, and constants imply that D is actually a linear map.

From this perspective, it's natural to ask about the kernel and image of D. The kernel certainly contains the constant functions, but is there anything else?

Theorem (Mean value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that the restriction $f|_{(a,b)} : (a, b) \to \mathbb{R}$ is differentiable. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Proof. We will prove the result in two cases. First assume that $f(a) = f(b)$. Then we want to find $c \in (a, b)$ such that $f'(c) = 0$. f is continuous on $[a, b]$, so by the min/max value theorem there exist $c_1, c_2 \in [a, b]$ such that $f(c_1) \le f(x) \le f(c_2)$ for all $x \in [a, b]$.

Suppose c_1 and c_2 are both at the endpoints. Then f is constant, and so it has derivative $f'(x) = 0$ everywhere.

Suppose at least one of c_1, c_2 in in (a, b). Without loss of generality say $c = c_2 \in (a, b)$ for $f(x) \le f(c)$ for all $x \in [a, b]$. The claim is that $f'(c) = 0$. Compute left and right limits for

$$\lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \le 0$$

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \ge 0$$

However, we know that this limit exists since f is differentiable by assumption, so both limits must be 0. This implies $f'(c) = 0$. This completes the proof in the case where $f(a) = f(b)$.

For the general case, consider the function

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

Then g satisfies the first case, so $g'(c) = 0$ for some $c \in (a, b)$. This proves that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(Check the details.) $\square$

Now we claim that the kernel of D is precisely the constant functions. Suppose $f'(x) = 0$ for all $x \in \mathbb{R}$. Apply the mean value theorem to $f|_{[0,a]}[0, a] \to \mathbb{R}$. There exists some $c \in (0, a)$ such that

$$0 = f'(c) = \frac{f(a) - f(0)}{a}$$

This implies $f(a) = f(0)$. Since a is arbitrary, this means that f is a constant function.

The image of D is a more difficult question that we will touch on later.

2/20/2019 - Polynomial approximation, second derivative test

Derivative interpretations

Last time we defined the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

which is the limit of the slope of the secant lines. The derivative captures the idea that, as we zoom in on a differentiable function, the function itself begins to look like a line.

We can use different notations for the derivative of a function.

- The notation $f'(a)$ was introduced by Langrange, around 1770.
- The notation $Df(a)$ emphasizes the fact that the derivative is a differential operator on the vector space of smooth functions.
- The notation $\frac{df}{dx}(a)$ was introduced by Leibniz, which he defined as the the value

$$\frac{f(a + dx) - f(a)}{dx}$$

when dx is 'infinitely small.' (This was before the limit was invented!)

If $f : (a, b) \to \mathbb{R}$ is differentiable, then we can view the derivative $f' : (a, b) \to \mathbb{R}$ as a function as well.

Definition. *If $f' : (a, b) \to \mathbb{R}$ as a function itself is continuous, then the function f is* ***continuously differentiable*** *and f is C^1. More generally, if $f', f'', \ldots, f^{(k)}$ all exist and are continuous, then f is C^k. Furthermore, if $f^{(k)}$ exist for all k, then f is* ***smooth****.*

Question: What does f' tell us about f?

Corollary. *If $f'(x) = 0$ for all x, then f is a constant function.*

Corollary. *Let f be C^1. Then if $f'(c) > 0$, f is increasing near c (meaning $x < y$ implies $f(x) < f(y)$. Similarly, if $f'(c) < 0$, f is decreasing near c.*

We'll use the mean value theorem, which says that if $f : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Since $f'(c) > 0$ and f' is continuous (as f is C^1), we know that f' is positive near c, namely that there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta)$ implies $f'(x) > 0$. Take points x and y such that

$$c - \delta < x < y < c + \delta$$

The mean value theorem for f on $[x, y]$ implies that there exists $c \in (x, y)$ such that

$$f(y) - f(x) = \underbrace{f'(c)}_{>0} \cdot \underbrace{(x, y)}_{>0} > 0$$

So we have shown that $x < y$ implies $f(x) < f(y)$. $\square$

Examples

- The function $f(x) = \sqrt{x}$ has derivative

  $$f'(x) = \frac{1}{2\sqrt{x}}$$

  We can show this formally by observing that if we define

  $$g(x) = f(x) \cdot f(x) = x$$

  then the product rule yields

  $$1 = g'(x) = 2f(x)f'(x)$$

  This implies

  $$f'(x) = \frac{1}{2f(x)}$$

  Note that to apply the product rule as we just did, we need to know that the functions involved are differentiable. So we really should compute the limit

  $$\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

  to determine the derivative of $\sqrt{x}$.

Proof of the product rule

Proof. If we have two differentiable functions $h, k : \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned}
(hk)'(a) &= \lim_{x \to a} \frac{h(x)k(x) - h(a)k(a)}{x - a} \\
&= \lim_{x \to a} \frac{h(x)k(x) - h(a)k(x)}{x - a} + \frac{h(a)k(x) - h(a)k(a)}{x - a} \\
&= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \cdot k(x) + h(a)\frac{k(x) = k(a)}{x - a} \\
&= h'(a)k(a) + h(a)k'(a)
\end{aligned}$$

where we are using the fact that both h and k are differentiable to simplify to the last line. $\square$

If we replace $f'(c) > 0$ with $f'(c) \geq 0$, we *cannot* assume that f is constant near c. For example, consider the functions $f(x) = x^2$ and $f(x) = x^3$, which both have derivative 0 at 0, but neither are constant.

Definition. *If $f'(c) = 0$, then c is a* ***critical point*** *of f.*

We showed last time that if c is a local minimum or maximum of a differentiable function f, then c is a critical point of f. We examined the left and right limits of

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

However, if c is a critical point of f, it is not necessarily true that f has a local minimum or maximum at c. For example, take the function $f(x) = x^3$.

Question: How can we use derivatives to describe the behavior of a function near a critical point?

Noteworthy/cautionary examples

Examples

- Define the function

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f is continuous. At $x = 0$, we can see this by examining

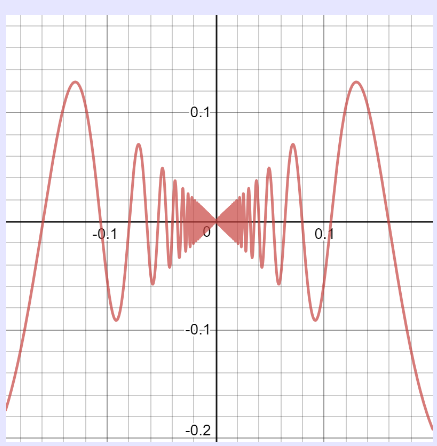
$$\lim_{x \to 0} x \sin(1/x)$$

Since $|\sin(1/x)| \leq 1$, the claim

$$\lim_{x \to 0} x \sin(1/x) = 0$$

follows from the squeeze theorem (squeeze this function between the functions x and $-x$). However, f is not differentiable at 0, since

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \sin(1/x)$$

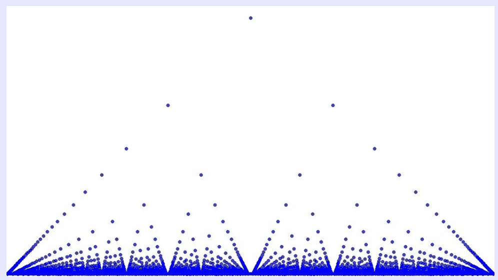


f is continuous on $[0,1]$ and differentiable on $(0,1)$ so we can apply the mean value theorem.

If we replace $x\sin(1/x)$ with $x^m \sin(1/x)$, we get examples that are C^r but not C^{r+1} (where r is related to m).

- Thomae's function $f : [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} \frac{1}{b} & \text{if } x = \frac{a}{b} \text{ in lowest terms} \\ 0 & x \text{ is irrational} \end{cases}$$



f is not continuous at each rational $a/b \in \mathbb{Q}$ (we can choose a sequence (x_n) that converges to x with x_n irrational). However, the surprising result is that f is continuous at all irrational numbers $x \notin \mathbb{Q}$. Fix some $\epsilon > 0$. Then there are only finitely many rational numbers a/b with $1/b > \epsilon$ (when $b < 1/\epsilon$). Choose δ small enough to avoid these rational numbers a/b.

Polynomial approximations

Definition. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^k function, and fix some $a \in \mathbb{R}$. Then the **kth Taylor polynomial** of f at a is*

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

In the case where $k = 1$, we have

$$P_1 = f(a) + f'(a)(x-a)$$

This is the linear approximation of the function f. In general, $P_k(x)$ is designed so that we have $P_k^{(i)}(a) = f^{(i)}(a)$ for $i \leq k$.

Theorem. *(Approximation theorem). Fix k, $f : \mathbb{R} \to \mathbb{R}$, and let $P = P_k$ be the kth Taylor polynomial. Then*

1. *P approximates f to order k at a. If we set $R(h) = f(a+h) - P(a+h)$, then*

$$\lim_{h \to 0} \frac{R(h)}{h^k} = 0$$

2. *P is the unique polynomial of degree less than or equal to k that satisfies the above condition.*

3. If f is actually C^{k+1}, then for each $h > 0$ there exists $c \in (a, a+h)$ such that

$$R(h) = \frac{f^{(k+1)}(c)}{(k+1)!} \cdot h^{k+1}$$

Intuitively, the theorem is saying that the Taylor polynomials are very good approximations for f near a. The function $R(h)$ should be viewed as the error term, which is the difference between our approximation and the function f.

$$f(a+h) = \underbrace{P(a+h)}_{\text{approximation}} + R(h)$$

The fact that the error term $R(h)$ has limit

$$\lim_{h \to 0} \frac{R(h)}{h^k} = 0$$

tells us that $R(h)$ is going to zero faster than h^k goes to zero, which means that $R(h)$ goes to zero extremely quickly. Polynomials are very easy to evaluate, graph, and understand, so it is important that we can approximate functions well with polynomials.

Corollary. *(Second derivative test). Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function, and let c be a critical point of f. If $f''(c) < 0$, then c is a local maximum of f, and if $f''(c) > 0$, then c is a local minimum of f.*

This is another example of how we can use the derivatives of a function to gain insight about its behavior.

Proof. Consider the 2nd Taylor polynomial

$$P(c+h) = f(c) + f'(c) \cdot h + \frac{f''(c)}{2} \cdot h^2$$

We assumed that c is a critical point of f, so the second term vanishes for

$$P(c+h) = f(c) + \frac{f''(c)}{2} \cdot h^2$$

This function is an upward or downward facing parabola. By the approximation theorem, we know that the limit

$$\begin{aligned}
\lim_{h \to 0} \frac{f(c+h) - P(c+h)}{h^2} &= \lim_{h \to 0} \frac{f(c+h) - f(c) - (f''(c)/2)h^2}{h^2} \\
&= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h^2} - \frac{f''(c)}{2}
\end{aligned}$$

should equal 0. In other words, we should have

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h^2} = \frac{f''(c)}{2}$$

Say $f''(c) > 0$. So when h is sufficiently small (in some open interval of width δ), we must have

$$\frac{f(c+h) - f(c)}{h^2} > 0$$

which implies that $f(c+h) - f(c) > 0$, so $f(c) < f(c+h)$. This means that c is a local minimum, as $f(c) < f(c+h)$ whenever $h \in (c - \delta, c + \delta)$. The case of a local maximum is similar. □

2/25/2019 - Integrability, Riemann integral

Polynomial approximations

Recall that if we have a C^k function $f : \mathbb{R} \to \mathbb{R}$, the kth order Taylor polynomial at a is defined

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}x^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The approximation theorem tells us

1. These polynomials are good approximations for f at a, namely that
$$\lim_{h \to 0} \frac{f(a+h) - P_k(a+h)}{h^k} = 0$$
2. These polynomials are unique. If $Q \in \mathrm{Poly}_k(\mathbb{R})$ is another polynomial that satisfies
$$\lim_{h \to 0} \frac{f(a+h) - Q(a+h)}{h^k} = 0$$
then $Q = P_k$.
3. If $f^{(k+1)}$ exists, then for $h \neq 0$ there exist $\theta \in (0, h)$ so that
$$R(h) = f(a+h) - P_k(a+h) = \frac{f^{(k+1)}(\theta)}{(k+1)!}h^{k+1}$$

Application of Taylor polynomials

We will approximation $\sin 1$. It's hard to imagine how we can get a grip on this one. Take $f(x) = \sin(x)$. The kth derivatives of sin are

$$\begin{cases} f(x) = \sin(x) & f(0) = 0 \\ f'(x) = \cos(x) & f'(0) = 1 \\ f''(x) = -\sin(x) & f''(0) = 0 \\ f'''(x) = -\cos(x) & f'''(0) = -1 \end{cases}$$

So we can compute

$$P_k(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5} + \ldots$$

The first few approximations of $\sin(1)$ with Taylor polynomials are

$$\begin{cases} P_1(1) = 1 \\ P_2(1) = 1 \\ P_3(1) = 1 - \frac{1}{6} \\ \dots \end{cases}$$

How far do we need to go to get $\sin(1)$ to 3 decimal places? The theorem implies

$$|R_k(1)| = \left| \frac{f^{k+1}(\theta)}{(k+1)!} 1^{k+1} \right| \leq \frac{1}{(k+1)!}$$

If we take $k = 7$, then we are bounding the error by

$$\frac{1}{7!} \approx 0.000024$$

Evaluating the 7th order Taylor polynomial yields

$$P_7(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} = \frac{4241}{5040} = 0.841468$$

$$\sin(1) \approx 0.841471$$

We will now prove the approximation theorem.

Proof. Let $k = 1$. Then we have

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - P_1(a+h)}{h} &= \lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} \\ &= \left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) - f'(a) \\ &= 0 \end{aligned}$$

since f is differentiable at a. Now let $k = 2$, which requires a different argument (the higher order cases follow similarly). Note that

$$R(h) = f(a+h) - P_2(a+h)$$

is C^2 (since f is C^2 and the polynomial P_2 is smooth) and $R(0) = 0$, $R'(0) = 0$, and $R''(0) = 0$ by construction of P_2 (since P_2 has the same 1st and 2nd derivatives as f). Then we have

$$\begin{aligned} \left| \frac{R(h)}{h^2} \right| &= \left| \frac{R(h) - R(0)}{h^2} \right| \\ &= \left| \frac{R'(t) \cdot h}{h^2} \right| \\ &= \left| \frac{R'(t)}{h} \right| \\ &\leq \left| \frac{R'(t)}{t} \right| \end{aligned}$$

where the second line is using the mean value theorem on $(0, h)$, and the final inequality follows because $t \leq h$. This expression goes to 0 as t goes to zero, since $R''(0) = 0$. This shows

$$\lim_{h \to 0} \frac{R(h)}{h^2} = 0$$

When $k \geq 3$, the proof works similarly. Iteratively apply the mean value theorem and repeat this argument to bring the exponenent of the denominator down, and then use the fact that R is C^k and all of its derivatives are 0.

To prove the second part of the theorem, assume that P and Q both approximate f to order k. Then

$$\begin{aligned} \lim_{h \to 0} \frac{P(a+h) - Q(a+h)}{h^k} &= \lim_{h \to 0} \frac{P(a+h) - f(a+h)}{h^k} + \frac{f(a+h) - Q(a+h)}{h^k} \\ &= 0 + 0 \end{aligned}$$

So P and Q approximate each other to order k. Write

$$P(x) = p_k x^k + \ldots + p_1 x + p_0$$
$$Q(x) = q_k x^k + \ldots + q_1 x + q_0$$

Then

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{P(a+h) - Q(a+h)}{h^k} \\ &= \lim_{h \to 0} \left(\frac{(p_k - q_k)(a+h)^k}{h^k} + \ldots + \frac{(p_1 - q_1)(a+h)}{h^k} + \frac{p_0 - q_0}{h^k} \right) \end{aligned}$$

This implies that $p_i - q_i$ must be zero for all i.

Perhaps the terms in this sum indeed go to infinity, but simply cancel each other out?

This can be patched by inducting on the degree of the polynomials P and Q. When $k = 0$, the uniqueness of P is equivalent to the uniqueness of the derivative. For $k > 0$, consider the truncated polynomials P' and Q' that consist of the first k terms of P and Q (excluding the x^k term). P' and Q' are $(k-1)$th order approximations of f, so by the inductive hypothesis they are equal. Applying the argument from above then yields $p_k = q_k$, so $P = Q$ as desired. Email me if you have questions.

To prove the last part of the theorem, we'll restrict our attention to $k = 1$. Say $h > 0$. We want to find $\theta \in (0, h)$ such that

$$\frac{f''(\theta)}{2} h^2 = R(h)$$
$$f''(\theta) = \frac{2R(h)}{h^2}$$

The idea will be to apply Rolle's theorem[4] to the cleverly chosen function

$$g(t) = R(t) - \frac{R(h)}{h^2} \cdot t^2$$

[4] A special case of the mean value theorem. If $g(a) = g(b)$, there exists some $c \in (a, b)$ such that $g'(c) = 0$.

We know $g(0) = 0 = g(h)$, so by Rolle's theorem there exists $s \in (0, h)$ such that

$$\begin{aligned} 0 &= g'(s) \\ &= R'(s) - \frac{2R(h)}{h^2} \cdot s \end{aligned}$$

Also $g'(0) = 0 = g'(s)$, so by Rolle's theorem (again!) there exists $s' \in (0, s)$ such that

$$\begin{aligned} 0 &= g''(s') \\ &= R''(s') - \frac{2R(h)}{h^2} \end{aligned}$$

We have

$$\begin{aligned} R(h) &= f(a+h) - f(a) - f'(a)h \\ R''(h) &= f'(a+h) \end{aligned}$$

Hence taking $\theta = s'$ yields

$$R(h) = \frac{f'(a+\theta)}{2} h^2$$

as desired. $\square$

Integration

The three big parts of calculus are differentiation, integration, and the fundamental theorem of calculus. We will now shift gears to integration.

Archimedes - 'Measurement of Circle'

Archimedes demonstrated two theorems

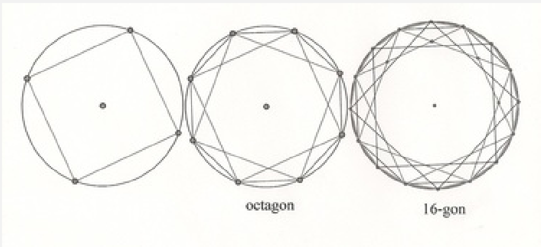
Theorem. *A circle C of radius r and circumference c has area $\frac{cr}{2}$.*

If we define π by $c = 2\pi r$, then the area is given by the familiar formula πr^2.

Theorem. *We have*

$$\frac{223}{71} < \pi < \frac{22}{7}$$

Proof. The proof of the first theorem is given by the 'method of exhaustion.' The idea is to approximate C by inscribed polygons.



If P_n is the regular n-sided inscribed polygon in C, let q be the perimeter of P_n and h be the length of the segment from the middle of a side of P_n and the center of the circle.

Note that the area of P_n is given $\frac{qh}{2}$, and that visually we have $\frac{cr}{2} \geq \frac{qh}{2}$. The claim (which will be proved below) is that the area of C is the supremum of the areas of P_n. From this it follows that the area of C is less than or equal to $\frac{cr}{2}$.

The reverse inequality is proved by similarly considering circumscribed polygons.

To prove the claim, we know that the area of C is greater than or equal to the supremum of the areas of P_n. To show equality we'll make

$$E_n = \text{Area}(C) - \text{Area}(P_n)$$

arbitrarily small. In fact,

$$E_{2n} \leq \frac{1}{2} E_n$$

To justify this, you should draw a picture that shows why doubling the number of sides more the halves the error of the approximation. $\square$

To approximate π in the second theorem, we just have to compute the perimeters of P_n. For example, the perimeter of P_6 is 6, so this tells us $\pi > 3$. There is a recursive formula for q_{2n} in terms of q_n. If

$$q_n = n \cdot s_n$$

where s_n is the side length of P_n, then we have

$$s_{2n}^2 = 2 - \sqrt{4 - s_n^2}$$

(Prove this with the Pythagorean theorem.) So we can easily compute

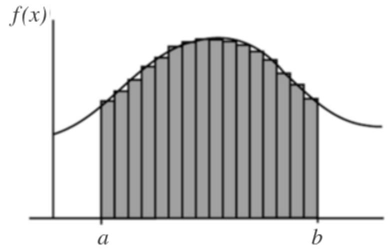
$$\begin{cases} s_6 = 1 \\ s_{12} = 2 - \sqrt{3} \\ s_{24} = 2 - \sqrt{2 + \sqrt{3}} \\ s_{48} = 2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}} \\ s_{96} = 2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}} \end{cases}$$

Archimedes approximated s_{96} to get a lower bound $\pi > \frac{223}{71}$. This is method of *extreme* exhaustion.

We will define the integral with a similar idea. Define the volume of a rectangle in $\mathbb{R}^n$ by

$$\text{vol}\big([a_1, b_1] \times \ldots \times [a_n, b_n]\big) = \prod_i^n (b_i - a_i)$$

We will then compute volumes of more complicated shapes by exhaustion.



To make this precise, we will need a few definitions.

Definition. *A **partition** P of $[a,b]$ is a finite subset $P \subset [a,b]$ that contains the endpoints. More generally, a partition of $Q = [a_1, b_1] \times \ldots \times [a_n, b_n]$ is a tuple $P(P_1, \ldots, P_n)$, where P_i is a partition of $[a_i, b_i]$.*

A partition P of a rectangle Q decomposes Q into subrectangles.

Definition. *Given a function $f : Q \to \mathbb{R}$ that is bounded, a partition P of Q, and a subrectangle $R \subset Q$, define*

$$m_R = \inf\{f(x) : x \in R\}$$

$$M_R = \sup\{f(x) : x \in R\}$$

*The **lower/upper sums of P** are*

$$L(f, P) = \sum_R m_R \cdot vol(R)$$

$$U(f, P) = \sum_R M_R \cdot vol(R)$$

Definition. *The **lower/upper integral** of a bounded function f is defined*

$$\underline{\int}_Q f = \sup\{L(f, P) : P \text{ a partition}\}$$

$$\overline{\int}_Q f = \inf\{U(f, P) : P \text{ a partition}\}$$

Definition. *A bounded function f is **integrable** if*

$$\underline{\int}_Q f = \overline{\int}_Q f$$

2/27/2019 - Integrability criteria, fundamental theorem of calculus, measure

The integral

Last time we defined the integral of a bounded function $f : Q \to \mathbb{R}$ defined on a closed rectangle $Q \subset \mathbb{R}^n$. We said f is integrable if

$$\underline{\int}_Q f = \overline{\int}_Q f$$

where the lower/upper integrals are defined as

$$\underline{\int}_Q f = \sup\{L(f,P) : P \text{ a partition}\}$$

$$\overline{\int}_Q f = \inf\{U(f,P) : P \text{ a partition}\}$$

where the lower/upper sums of a partition P are

$$L(f,P) = \sum_{R \text{ a subrectangle}} m_R \cdot \operatorname{vol}(R)$$

$$U(f,P) = \sum_{R \text{ a subrectangle}} M_R \cdot \operatorname{vol}(R)$$

and

$$m_R = \inf\{f(x) : x \in R\}$$
$$M_R = \sup\{f(x) : x \in R\}$$

Examples

- Define $f : [0,1] \to \mathbb{R}$ by

  $$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

  For any P, we have

  $$U(f,P) = \sum_R M_R \cdot \operatorname{vol}(R)$$
  $$= 1$$

  Since the rationals are dense in $[0,1]$, there will always be some rational in R, which forces $M_R = 1$. Similarly

  $$L(f,P) = 0$$

  for all partitions P, which means that f is not integrable.

- Define $f : [0,1] \to \mathbb{R}$ on $[0,1]$. We claim
$$\int_{[0,1]} f = \frac{1}{2}$$
This matches our intuition, as the integral should measure the area of the triangle under the graph of f from 0 to 1.

To prove this, define the partition
$$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$$
Then we have
$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2} \sum_{i=1}^{n} i \\ &= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \\ &= \frac{1}{2}(1 + \frac{1}{n}) \end{aligned}$$
The lower sum is similarly
$$\begin{aligned} L(f, P_n) &= \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n^2} \\ &= \frac{1}{2}(1 - \frac{1}{n}) \end{aligned}$$
Then we have
$$\begin{aligned} \frac{1}{2} &= \sup\{L(f, P_n)\} \\ &\leq \sup\{L(f, P)\} \\ &\leq \inf\{U(f, P)\} \\ &\leq \inf\{U(f, P_n)\} = \frac{1}{2} \end{aligned}$$
which proves that
$$\underline{\int}_Q f = \overline{\int}_Q f = \frac{1}{2}$$
- Define $f : [0,1] \to \mathbb{R}$ on $[0,1]$ by
$$\begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$$

f is indeed integrable. To prove this, define the partition

$$P_n = \{0, \frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}, 1\}$$

Then

$$U(f, P_n) = 1 \cdot \frac{2}{n} = \frac{2}{n}$$

By taking n, the upper and lower sums both converge to 0, so f is integrable with integral 0.

In general, the third example from above should convince you that if f is not continuous at only finitely many points, then f is integrable. We will explore the question of precisely when a function is integrable later.

Question: In the definition of the integral, where did we use the fact that f is bounded?

Answer: To define m_R and M_R, we need to know that f is bounded (for the supremum and infinum to exist).

Question: Why are $\underline{\int}_Q f$ and $\overline{\int}_Q f$ always defined?

Answer: All the lower sums are bounded by any upper sum, and vice versa. See the lemma below.

Lemma. *For any partitions P and P'*

$$L(f, P) \leq U(f, P')$$

In particular, $\{L(f, P)\}$ are bounded above by any $U(f, P')$.

Definition. *Let $P = (P_1, \ldots, P_n)$ and $\hat{P} = (\hat{P}_1, \ldots, \hat{P}_n)$ be partitions of Q. $\hat{P}$ is a* ***refinement*** *of P if $P_i \subset \hat{P}_i$ for all i.*

$\hat{P}$ is a refinement because the subrectangles are smaller: each subrectangle of $\hat{P}$ is contained in a subrectangle of P.

Proof. There are two important facts about partitions we will use.

1. If $\hat{P}$ is a refinement of P, then

$$L(f, P) \leq L(f, \hat{P}) \tag{1}$$
$$U(f, P) \geq U(f, \hat{P}) \tag{2}$$

Inequality (1) follows because if $\hat{R} \subset R$, then $m_{\hat{R}} \geq m_R$, as we are looking at the infinum of f on a set that contains fewer points. Inequality (2) follows from a similar reason. So refinements make the lower sums increase and the upper sums decrease.

2. Any two partitions P and P' have a ***common refinement*** given by

$$P'' = (P_1 \cup P_1', \ldots, P_n \cup P_n')$$

Given P and P', let P'' be their common refinement. Then we have

$$\begin{aligned} L(f,P) &\leq L(f,P'') \\ &\leq U(f,P'') \\ &\leq U(f,P') \end{aligned}$$

which is the inequality we were aiming for. $\square$

There are two components of the theory of integration. The first part seeks to answer which functions are integrable. The second part develops techniques to compute integrals without resorting to the definitions (namely with the fundamental theorem of calculus).

Integrability criteria

Theorem. *If $f : Q \to \mathbb{R}$ is continuous, then f is integrable.*

Any important observation will help us here. If for every $\epsilon_0 > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$, then f is integrable. It's easy to see the contrapositive: if f is *not* integrable, then the infininum of $\{U(f,P)\}$ and the supremum of $\{L(f,P)\}$ differ. Take ϵ to be the distance between them. Then there is no partition such that $U(f,P) - L(f,P) < \epsilon$, since this would contradict the infinum/supremum on the upper/lower sums.

Recall that if f is continuous on Q (a closed rectangle), then f is uniformly continuous. So for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Proof. Fix $\epsilon > 0$. By the above remark, we want to find P such that $U(f,P) - L(f,P) < \epsilon$. In other words, we want to make the quantity

$$U(f,P) - L(f,P) = \sum_R (M_R - m_R) \cdot \mathrm{vol}(R)$$

By the uniform continuity of f, we can choose $\delta > 0$ such that $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \frac{\epsilon}{\mathrm{vol}(Q)}$$

Then choose a partition P fine enough so that all the subrectangles have diameter smaller than δ. Then

$$\sum_R (M_R - m_R) \cdot \mathrm{vol}(Q) < \sum_R \frac{\epsilon}{\mathrm{vol}(Q)} \cdot \mathrm{vol}(Q) = \epsilon$$

Then f is integrable, as desired. $\square$

Examples

- Define the set
$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$$
Define the function $\chi_B : Q \to \mathbb{R}$ by
$$\chi_B(x, y) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$
The integral $\int_Q \chi_B$ give us the area of the circle (as the graph of χ_B in three dimensions is a cyclinder of length 1).

  χ_B is discontinuous at all points in the boundary of B, which consists of the circle of radius 1. We would like to be able to compute integrals like this.

Theorem. *If $B \subset \mathbb{R}^n$ is a subset, then χ_B is the **characteristic function** of B and is defined*
$$\chi_B = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$
χ_B is integrable if and only if the boundary of B has measure 0.

We will come back to this theorem and define the different parts of the statement.

The fundamental theorem of calculus

If $a < b$, we sometimes write
$$\int_{[a,b]} f = \int_a^b f = \int_a^b f(x)\, dx$$
dx is just a symbol write now. As a convention, we take
$$\int_a^b f = -\int_b^a f$$
Theorem. *(Fundamental theorem of calculus). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then*

1. *The function F defined*
$$F(t) = \int_a^t f(x)\, dx$$
is differentiable on (a, b), and its derivative F' is given by f.
2. *If $g : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) with $g' = f$, then*
$$\int_a^b f = g(b) - g(a)$$

The second part of the theorem tells us we can compute the integral of any function for which we know an antiderivative. The first part of the statement says that any function has an antiderivative (if we just take $g = F$).

Examples

- We have
$$\int_0^1 x^2 - x^3 = g(1) - g(0) = \frac{1}{3} - \frac{1}{4}$$
for $g(x) = \frac{1}{3}x^3 - \frac{1}{4}x^4$.

We should believe that the function F defined above is differentiable, since the difference quotient

$$\frac{F(t+h) - F(t)}{h} \approx \frac{h \cdot f(t)}{h} = f(t)$$

since f is continuous (draw a picture to make this argument clear).

We will use the integral to *define* the area of a subset of $\mathbb{R}^n$ by successively performing better approximations of the subset with rectangles. This idea agrees with our intuition of the area of simple shapes, and it will turn out to be the correct definition to take.

Measure

Definition. *A subset $B \subset \mathbb{R}^n$ has* ***measure 0*** *if for all $\epsilon > 0$, there exist countably many rectangles $Q_1, Q_2, \ldots$ (or balls) that cover B so that*

$$B \subset \bigcup_{i=1}^{\infty} Q_i$$

$$\sum_{i=1}^{\infty} vol(Q_i) < \epsilon$$

Examples

- If $B \subset \mathbb{R}^n$ is finite, B has measure 0. Cover each point $x \in B$ with a ball of total volume less than $\epsilon/|B|$.
- $\mathbb{Q} \subset \mathbb{R}$ has measure 0. Since $\mathbb{Q}$ is countable, write $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Take
$$Q_i = \big(q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i}\big)$$
The $\{Q_i : i \in \mathbb{N}\}$ rectangles cover $\mathbb{Q}$, and
$$\sum_{i=1}^{\infty} \text{vol}(Q_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$
using the formula for the sum of a geometric series.

3/4/2019 - Integrability and measure, Cavalieri and Fubini

Integrability and measure

Recall that a subset $B \subset \mathbb{R}^n$ has measure 0 if for all $\epsilon > 0$, there exist open rectangles $Q_1, Q_2, \ldots$ that cover B such that

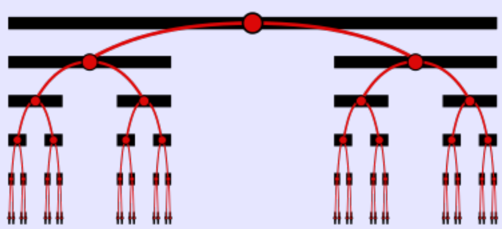
$$\sum \text{vol}(q_i) < \epsilon$$

We saw that finite sets, countable sets, and countable unions of measure 0 sets are all measure 0. The idea with the second two is to cover the set with open rectangles whose area is shrinking like the terms of a geometric series, and hence converges to a value less than ϵ.

The Cantor set

Let $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 3]$, and continue to inductively define C_n by removing the middle third of each subinterval. Then the Cantor set is given by

$$C = \bigcap_{n=1}^{\infty} C_n$$



C has measure 0. Observe that C_0 has volume 1, C_1 has volume $2/3$, C_2 has volume $4/9$, and so on. So C_n has volume $2^n/3^n$. Since

$$\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

we can always consider n large enough so that it is possible to cover C with open sets of arbitrary volume.

Note that although C has measure 0, it is uncountable. We can represent C as the ternary decimal expansions $0.a_1a_2\ldots$ that don't end in repeating the value 2 and that don't contain any 1.

We can use measure to characterize integrable functions.

Theorem (Riemann-Lebesgue Theorem). *Let $f : Q \to \mathbb{R}$ be a bounded function, with $Q \subset \mathbb{R}^n$ a closed rectangle. Define*

$$B_f = \{x \in Q : f \text{ is not continuous at } x\}$$

Then f is integrable if and only if B_f has measure 0.

The theorem from last time is a particular instance of this more general result (see the below example). We won't prove this now, but some aspects of it are on the homework.

Examples

- If f is continuous, then $B_f = \varnothing$, so $\int_Q f$ exists. We proved last time this was integrable using uniform continuity, but it now follows directly from the theorem.
- Let $C \subset \mathbb{R}^n$ be bounded at $C \subset Q$ with Q a closed rectangle. The characteristic function of C is
$$\chi_C(x) = \begin{cases} 1 & x \in C \\ 0 & \text{otherwise} \end{cases}$$
Then $B_{\chi_C} = \text{bd}(C)$, so χ_C is integrable if and only if $\text{bd}(C)$ has measure 0.
- Define
$$f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$
$$g(x) = \chi_{C\cap[0,1]} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$
$B_f = \{1/2\}$ and has measure 0, so f is integrable. But $B_g = \text{bd}(\mathbb{Q} \cap [0,1]) = [0,1]$ (as every neighborhood of any $x \in [0,1]$ contains both rational and irrational points), so g is not integrable.
- Let $C = B_1(0) \subset \mathbb{R}^3$. Then the boundary of B_1 is the hollow sphere of radius 1, and has measure 0 (by the homework). Then
$$\text{vol}(C) = \int_{[-1,1]} \chi_C$$
exists, and we will return to the question of evaluating it later.
- Recall that Thomae's function is defined
$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & \text{otherwise} \end{cases}$$
We know $B_T = \mathbb{Q} \cap [0,1]$, which are countable. Then B_T has measure 0, so f is integrable. Note the difference between this example and g defined above. Thomae's function is integrable because it is in fact continuous at all irrational numbers.

We can also evaluate some of the integrals using the below result, which is left as an exercise.

Let $f : Q \to \mathbb{R}$ be a function, and suppose the set

$$\{x \in Q : f(x) \neq 0\}$$

has measure 0. If f is integrable, then

$$\int_Q f = 0$$

This claim is not too hard to prove, and it is a worthwhile exercise. This result implies that the integral of Thomae's function is 0.

The fundamental theorem of calculus

Theorem (Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then*

(i) $F(x) = \int_a^x f$ is differentiable, and $F'(x) = f(x)$.

(ii) If $g[a, b] \to \mathbb{R}$ is a differentiable such that $g'(x) = f(x)$ for all x, then

$$\int_a^b f = g(b) - g(a)$$

This theorem implies all of the familiar integration rules, such as substitution of variables and integration by parts.

Corollary (Substitution of variables)**.** *Let $f : [a, b] \to \mathbb{R}$, and suppose $u : [c, d] \to [a, b]$ is bijective and differentiable with $u(c) = a$ and $u(d) = b$. Then*

$$\int_a^b f(x)\, dx = \int_c^d f(u(y))u'(y)\, dy$$

Proof. If $F'(x) = f(x)$, then

$$\begin{aligned}(F \circ u)'(y) &= F'(u(y))u'(y)\\ &= f(u(y))u'(y)\end{aligned}$$

by the chain rule and the fundamental theorem of calculus. Then this yields

$$\begin{aligned}\int_c^d f(u(y))u'(y)\, dy &= (F \circ u)(d) - (F \circ u)(c)\\ &= F(b) - F(a)\\ &= \int_a^b f(x)\, dx\end{aligned}$$

$\square$

Corollary (Integration by parts)**.** *If* $f, g : [a,b] \to \mathbb{R}$ *are differentiable, then we have*

$$[fg]_a^b = \int_a^b (fg)' = \int_a^b f'g + fg'$$

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Examples

- We can compute

$$\begin{aligned}\int_0^{\pi/2} \cos^2(\theta)\, d\theta &= \int_0^{\pi/2} \underbrace{\cos\theta}_{f'}\underbrace{\cos\theta}_{g}\, d\theta \\ &= \left[\sin\theta\cos\theta\right]_0^{\pi/2} \sin\theta(-\sin\theta)\, d\theta \\ &= \int_0^{\pi/2} \sin^2\theta\, d\theta \\ &= \int_0^{\pi/2} 1\, d\theta - \cos^2\theta\, d\theta \\ &= \int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2} \cos^2\theta\, d\theta\end{aligned}$$

  using integration by parts. Hence

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{4}$$

- We can compute the area of the circle by integration

$$A = 2 \cdot \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

  Subsituting $x = \sin\theta$ and $dx = \cos\theta\, d\theta$ yields

$$\begin{aligned}A &= 2 \cdot \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta \\ &= 2 \cdot \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \\ &= 2 \cdot \left(2 \cdot \int_0^{\pi/2} \cos^2\theta\, d\theta\right) \\ &= \pi\end{aligned}$$

  as cosine is even and therefore symmetric about the y-axis.

Proof of the fundamental theorem of calculus. Let $f : [a,b] \to \mathbb{R}$ be a continuous function and define $F(x) = \int_a^x f$ as in the theorem. To prove the first part of the theorem, observe that we have

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h}$$

And

$$\frac{F(c+h) - F(c)}{h} = \frac{\int_a^{c+h} f - \int_a^c f}{h} = \frac{\int_c^{c+h} f}{h}$$

Say $h > 0$ (the other case is identical). Let m, M be the minimum and maximum of f on the closed interval $[c, c+h]$. Then

$$m \cdot h \leq \int_c^{c+h} f \leq M \cdot h$$

$$m \leq \frac{1}{h} \int_c^{c+h} f \leq M$$

You will prove this intuitive fact rigorously on the homework. As h goes to 0, m and M converge to $f(c)$, since f is continuous by assumption.

To prove the second part of the theorem, let g be an antiderivative of f. Then both $g' = f$ and $F' = f$. This implies $(g - F)' = 0$, so $g - F = c$ for some constant c. We can compute this constant

$$c = g(a) - F(a) = g(a) - 0$$

Then

$$g(a) = g(b) - F(b) = g(b) - \int_a^b f$$

$$\int_a^b f = g(b) - g(a)$$

as desired. $\square$

Examples

- Define $H(x) = \int_0^{x^2} f(t)\, dt$. Then we can view H as the composition $x \mapsto x^2 \mapsto \int_0^{x^2} f$. By substitution of variables, we have

  $$H(x) = \int_0^{x^2} f(t)\, dt = \int_0^x f(x^2) \cdot 2x\, dx$$

  Then the fundamental theorem of calculus implies $H'(x) = 2xf(x)$.

Fubini's Formula

We will introduce one more theorem that's extremely useful for evaluating integrals. We would like to be able to use the fundamental theorem of calculus, which tells us how to integrate over an interval in $\mathbb{R}$, to compute integrals in higher-dimensional settings. It works by cutting up the domain into 'slices' that are one-dimensional integrals and then performing a successive integrals.

Theorem (Fubini's Formula)**.** *Let* $Q = [a,b] \times [c,d]$ *and* $f : Q \to \mathbb{R}$ *be continuous. Then*

$$\int_Q f = \int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\, dy \right) dx$$

Examples

- We can use Fubini's theorem to compute the volume of $B_1(0) \subset \mathbb{R}^3$. Define the function $f : Q = [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \sqrt{1 - x^2 - y^2} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

  Then

$$\begin{aligned} \mathrm{vol}(B) &= 8 \cdot \int_Q f \\ &= 8 \cdot \int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\, dy \right) dx \\ &= 8 \cdot \int_{x=0}^{1} \frac{\pi}{4}(1 - x^2)\, dx \\ &= 2\pi \left[x - \frac{x^3}{3} \right]_0^1 \\ &= \frac{4\pi}{3} \end{aligned}$$

Theorem (Fubini's Theorem)**.** *Let* $Q = A \times B \subset \mathbb{R}^k \times \mathbb{R}^m$ *and* $f : Q \to \mathbb{R}$ *be continuous. Then*

$$\int_Q f = \int_A \left(\int_B f(x,y)\, dy \right) dx$$

The continuity assumption is important here. For example, Fubini's theorem fails for the function

$$f(x,y) = \begin{cases} 1 & x = \frac{1}{2}, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

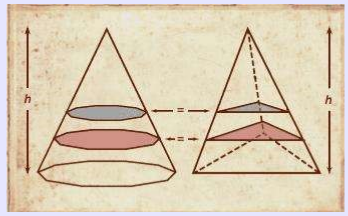
defined from $[0,1]^2$, as

$$\int_{y=0}^{1} f(\frac{1}{2}, y)\, dy$$

does not exist. It is possible to remedy this problem and further strengthen the theorem.

Application

Theorem (Cavalieri's Principle)**.** *Shapes with cross sections of equal area have equal volume.*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3/6/2019 - Function spaces, ODEs

Fubini's theorem

Recall Fubini's theorem, which says if $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, then we can compute

$$\int_Q f = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\, dy\, dx$$

There is a version of the theorem for when f is discontinuous but still integrable. See Pugh for the details. We will prove the continuous case now.

Proof. Define $I(x) = \int_c^d f(x, y)\, dy$. We want to show

$$\int_Q f = \int_a^b I$$

Fix a partition $P = (P_1, P_2)$ of Q. The main claim of the proof will be that

$$L(f, P) \le L(I, P_1) \le U(I, P_1) \le U(f, P)$$

Since f is integrable, this implies that I is integrable and $\int_Q f = \int_a^b I$.

Let's show $L(f, P) \le L(I, P_1)$. If S is a subinterval of P_1 and T is a subinterval of P_2 we have

$$\begin{aligned} L(f, P) &= \sum_{S \times T} \inf\{f(x, y) : (x, y) \in S \times T\} \cdot \mathrm{vol}(S \times T) \\ &= \sum_S \underbrace{\left(\sum_T \inf\{f(x, y) : (x, y) \in S \times T\} \cdot \mathrm{vol}(T) \right)}_{(*)} \cdot \mathrm{vol}(S) \end{aligned}$$

and

$$L(I, P_1) = \sum_S \underbrace{\inf\{I(x) : x \in S\}}_{(**)} \cdot \mathrm{vol}(S)$$

Then it's evident that if we want to show $L(f, P) \le L(I, P_1)$, we should prove that $(*)$ is less than or equal to $(**)$. Fix some $x \in S$, then

$$\begin{aligned} I(x) &= \int_c^d f(x, y)\, dy \\ &\ge L(f(x,), P_2) \\ &= \sum_T \inf\{f(x, y) : y \in T\} \cdot \mathrm{vol}(T) \\ &\ge \sum_T \inf\{f(x, y) : (x, y) \in S \times T\} \cdot \mathrm{vol}(T) = (*) \end{aligned}$$

since if we take the infinum over the larger set $S \times T$, the result is smaller than the infinum over the slice $\{x\} \times T$. So for each fixed $x \in S$, $I(x)$ is greater than or equal to $(*)$. Then $\inf\{I(X) : x \in S\}$, which is $(**)$, is greater than or equal to $(*)$.

The inequality $U(I, P_1) \le U(f, P)$ follows similarly, so this completes the proof. $\square$

This completes our discussion of calculus. We will develop many of these tools in the multivariable context, but first we will explore differential equations.

Differential equations

Definition. *Given a function $f(t,x)$, define a* ***differential equation***

$$x' = f(t,x)$$

A differential equation involves both a function and its derivatives. A ***solution*** *to this equation is a function $x(t)$ so that*

$$x'(t) = f(t, x(t))$$

for all t.

Examples

- Consider the function given by

  $$f(t,x) = \frac{-t}{x}$$

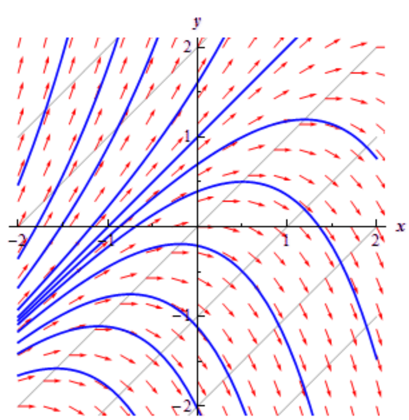
  Then $x' = -t/x$. The function

  $$x(t) = \sqrt{a^2 - t^2}$$

  is a solution to this differential equation, as

  $$x'(t) = \frac{-2t}{2\sqrt{a^2-t^2}} = \frac{-t}{\sqrt{a^2-t^2}} = \frac{-t}{x(t)}$$

Differential equations often model physical phenomena. We can think of $x(t)$ as the position of a particle at time t. Geometrically, we can view a differential equation by constructing a plot in the following way. At a point $(t,x) \in \mathbb{R}^2$, draw a line segment with slope $f(t,x)$.



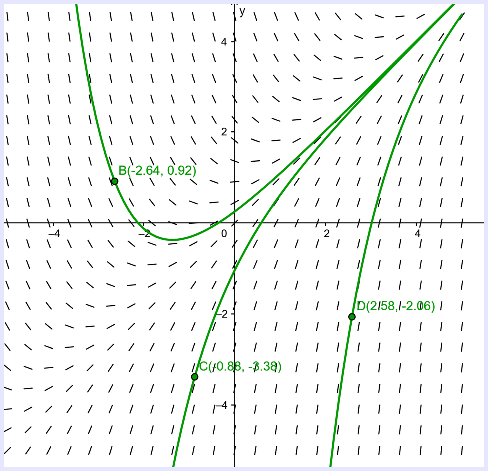
Then the graph of a solution $x(t)$ is tangent everywhere to the line field (which is the collection of points in $\mathbb{R}^2$ with their associated lines). This follows from what it means to be a solution and how we constructed this line field.

Examples

- Let $x' = 1 + t - x$. To plot the line field, for each $c \in \mathbb{R}$ draw the curve $f(t, x) = c$. Then draw the line segments of slope c along this curve.

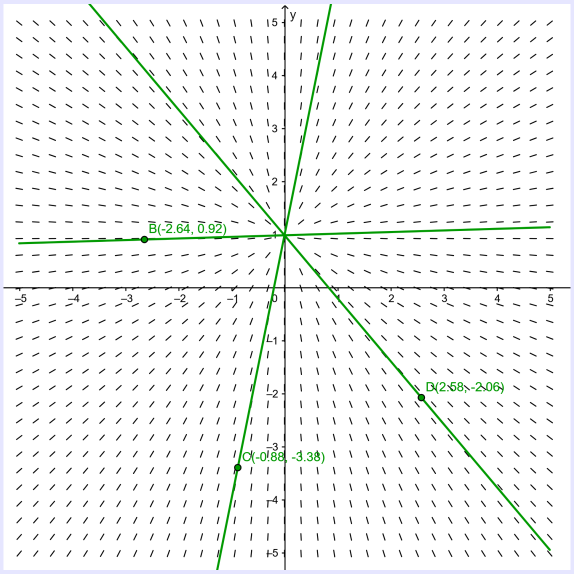
  Without being able to explicitly solve for $x(t)$, we can get a sense of their form by examining the graph of this differential equation.

  Plot a solution by choosing an ***initial condition*** $x(t_0) = x_0$. Then follow the line field, beginning at the point $(t_0, x_0) \in \mathbb{R}^2$ to find as solution. In this example, $x(t) = t$ is a solution.



  There are some questions we can ask about these solutions. For example, it seems like all of the solutions converge to the line $x(t) = t$ in some sense. What exactly is happening? Is it possible for a solution to 'cross' this line?

- Let $x' = \frac{x-1}{t}$. Then we have the following graph and solutions.



General Questions: Consider a differential equation $f(t, x)$ and intitial condition $x(t_0) = x_0$.

1. Is there a solution to
$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$
2. Is the solution unique (for a given initial condition)? In general the answer is no.

Examples

- For example, the equation $x' = \frac{x-1}{t}$ from above has no solutions when the initial condition is $x(0) = x_0 \neq 1$. Uniqueness fails for the initial condition $x(0) = 1$.

To answer these questions, we will develop the theory of function spaces.

Space of bounded functions

Definition. *The **space of bounded functions** is the vector space*

$$C_b = C_b([a, b], \mathbb{R}) = \{f : [a, b] \to \mathbb{R} \text{ bounded}\}$$

*C_b comes with a natural norm, called the **sup norm**, given by*

$$\begin{aligned} \|\cdot\| : C_b &\to [0, \infty) \\ f &\mapsto \|f\| = \sup\{|f(x)| : x \in [a, b]\} \end{aligned}$$

The sup norm satisfies the three essential properties of a norm:

1. Nondegeneracy, which means $\|f\| = 0$ if and only if $f = 0$.
2. Compatibility with multiplication, which means for scalars $c \in \mathbb{R}$ we have $\|cf\| = |c| \cdot \|f\|$.
3. Triangle inequality, which means $\|f + g\| \leq \|f\| + \|g\|$.

We can interpret $\|f - g\|$ as the 'distance' between two functions, where the distance between f and g is the largest distance between the graphs of the two functions.

The sup norm makes C_b into a **metric space**, which is just a set that has a distance function that is symmetric, nondegenerate, and satifies the triangle inequality. The important idea is that this allows us to employ *topological* concepts when talking about the function space C_b.

Definition. *A sequence of functions (f_n) with $f_n \in C_b$ **converges uniformly** to $f \in C_b$ if*

$$\lim_{n\to\infty} \|f_n - f\| = 0$$

Definition. *A sequence of functions (f_n) with $f_n \in C_b$ **converges pointwise** to $f \in C_b$ if*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all x.

Note that uniform convergence implies pointwise convergence, but the converse *does not* necessarily hold. A nice counterexample comes from taking

$$f_n = \chi_{[n,n+1]}$$

The f_n functions converge pointwise to 0, but they do not converge uniformly to anything.

Examples

- Define $f_n(x) = \frac{x}{n}$ on the interval $[0, 1]$. Then (f_n) converges uniformly to 0.
- Define $f_n(x) = x^n$ on $[0, 1]$. For $x < 1$,

  $$\lim_{n\to\infty} f_n(x) = 0$$

  For $x = 1$, $f_n(x) = 1$ for all n. Then (f_n) converges pointwise to the function

  $$f(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

  However, (f_n) do not converge uniformly. Since each f_n is continuous, for a small ϵ the function cannot jump from an ϵ neighborhood around 0 to an ϵ neighborhood around 1 without passing through the intermediary points. This is a special case of the below theorem.

Note that if a sequence converges pointwise to some function, then if they converge uniformly they must converge uniformly to that function. In general, uniform convergence preserves nice properties of functions like continuity and integrability.

Theorem. *Suppose the sequence (f_n) converges uniformly to a function f, where each f_n is continuous. Then f is continuous.*

Proof. To prove that f is continuous at $p \in [a, b]$, fix $\epsilon > 0$. Then the triangle inequality applied twice yields

$$|f(x) - f(p)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(p)| + |f_n(p) - f(p)|$$

The left and right terms are small by uniform continuity, and the middle term is small by the continuity of f_n. To make this prcise, choose n large enough so that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all x by uniform convergence. Choose $\delta > 0$ such that $|x - p| < \delta$ implies

$$|f_n(x) - f_n(p)| < \frac{\epsilon}{3}$$

Then $|x - p < \delta$ implies $|f(x) - f(p) < \epsilon$, as desired. $\square$

Corollary. *$C^0 \subset C_b$ is a closed subspace. In words, the space of continuous functions is closed in the bounded functions.*

Proof. This follows immediately from the previous theorem, since any converging sequence of functions in C^0 converges to a function in C^0. $\square$

Theorem. *C_b is a* ***complete*** *metric space. In other words, if (f_n) is a Cauchy sequence in C_b, then f_n converges to some $f \in C_b$.*

We will prove this next time.

Corollary. *The continuous functions $C^0 \subset C_b$ is also complete.*

Proof. Let (f_n) be a Cauchy sequence of continuous functions. C_b is complete, so (f_n) converges to some $f \in C_b$. Uniform convergence of continuous functions to f implies that f is continuous, as desired. $\square$

3/11/2019 - Function convergence, equicontinuity, Arzela-Ascoli theorem

ODE (ordinary differential equation) existence

Theorem (Peano's Theorem). *If $f(t,x)$ is continuous near (t_0, x_0), then the initial value problem*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

has a solution near t_0. Namely, there exists $\epsilon > 0$ and a function $\phi : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}$ such that $\phi(t_0) = x_0$ and $\phi'(t) = f(t, \phi(t))$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Examples

- We saw last time that the solutions of $x' = -t/x$ are of the form $\phi(t) = \sqrt{a^2 - x^2}$ (semicircles). This solution is only defined on $[-a, a]$, which is why the theorem only guarantees the existence of a local solution. It doesn't imply that there is always a solution defined on all of $\mathbb{R}$.

  The theorem also doesn't say anything about the case when $x(t_0) = 0$, since $f(t,x) = -t/x$ is not continuous at $(t_0, 0)$.

- Uniqueness of solutions is also not guaranteed. Consider the function

  $$f(t,x) = \begin{cases} \sqrt{x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

  Consider the solutions that begin at $(0,0)$. Then we can follow the slope field to find solutions. The obvious solution $\phi_1(t) = 0$, which is a horizontal line. But there's also another solution given by

  $$\phi_2(t) = \begin{cases} \frac{t^2}{4} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

There is a stronger theorem called *Picard's theorem* that guarantees the existence and uniqueness of a solution to a differential equation, assuming that the function f is *Lipschitz*. We won't speak about this, but you can see Pugh for details.

Uniform convergence

Recall that $C_b = C_b([a,b])$ is the vector space of bounded functions with a norm defined by

$$\|f\| = \sup\{|f(x)| : x \in [a,b]\}$$

This makes C_b into a metric space. We can talk about uniform convergence, which says that a sequence (f_n) converges to a function f if $\lim_{n\to\infty} \|f_n - f\| = 0$.

We proved last time that if a sequence of functions (f_n) that are continuous at p converges to f, then f is continuous at p as well. So uniform convergence preserves continuity. We can show that uniform convergence preserves other 'nice' properties of functions as well.

Theorem. *If a sequence of functions (f_n) converges uniformly to f and each f_n is integrable, then f is integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

In such a situation, we might say that 'limits and integrals commute,' since this equation is equivalent to

$$\int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n$$

Proof. To prove that f is integrable, note that by the Riemann-Lebesgue theorem, it suffices to show that

$$B_f = \{x \in [a,b] : f \text{ discontinuous at } x\}$$

has measure 0. We know that f_n is integrable, so B_{f_n} has measure 0. Then each f_n is continuous on $[a,b] \setminus B_{f_n}$. All of the f_n functions are continuous on

$$[a,b] \setminus \bigcup_k B_{f_k}$$

By the above theorem, this implies that f is continuous on $[a,b] \setminus \bigcup B_{f_k}$. $\bigcup B_{f_k}$ has measure 0, since it is the countable union of measure 0 sets. Therefore f is integrable.

To show that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$, observe that we have

$$\begin{aligned}\left|\int_a^b f - \int_a^b f_n\right| &= \left|\int_a^b f - f_n\right| \\ &\le \int_a^b |f(x) - f_n(x)|\, dx \\ &\le \|f - f_n\| \cdot (b-a)\end{aligned}$$

by Homework 4. As n goes to infinity, the quantity $\|f - f_n\|$ goes to 0, as desired. □

We also have the theorem from last time asserting the completeness of the function space C_b.

Theorem. *If (f_n) is a Cauchy sequence in C_b, then there exists $f \in C_b$ such that f_n converge uniformly to f.*

Proof. Since (f_n) is a Cauchy sequence, for all $\epsilon > 0$ there exists $N > 0$ such that $n, m > N$ implies $|f_n(x) - f_m(x)| < \epsilon$ for all x. Then for each particular $x \in [a,b]$, then sequence $(f_n(x))$ in $\mathbb{R}$ is Cauchy. $\mathbb{R}$ is complete, so each of these sequences converges. Then we can define

$$(x) = \lim_{n\to\infty} f_n(x)$$

So we know that the f_n functions converge pointwise to f, but we must show that they also converge uniformly. Fix $\epsilon > 0$. Then

$$|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|$$

We know the second term is small for all x because (f_n) is a Cauchy sequence, and we know the first term is small for each x when m is sufficiently large. Choose N such that $n, m > N$ implies $\|f_n - f_m\| < \epsilon/2$. For each x, choose $m_x > N$ so that $|f(x) - f_m(x)| < \epsilon/2$.

If $n > N$, then for each $x \in [a, b]$ we have

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_{m_x}(x)| + |f_{m_x}(x) - f_n(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

We must still show $f \in C_b$. Since $\|f_n - f\| < 1$, this implies $\|f\| \leq \|f_n\| + 1$ by the triangle inequality. So f is bounded, which completes the proof. □

Arzela-Ascoli theorem

We will now turn to the Bolzano-Weierstrass theorem for C_b. Recall that the Bolzano-Weierstrass theorem stated that if (x_n) is a bounded sequence in $\mathbb{R}$, then there exists $x \in \mathbb{R}$ and a subsequence (x_{n_k}) of (x_n) such that x_{n_k} converge to x.

Question: What condition on a sequence (f_n) in C_b guarantees a uniformly convergent subsequence?

The naive approach would be to similarly demand that (f_n) are bounded in norm (meaning $\|f_n\| \leq M$ for some $M > 0$), but this is not enough. For example, the sequence of functions defined by $f_n(x) = x^n$ on $[0, 1]$ is bounded in norm by 1, but there is no convergent subsequence. This is easy to see, since f_n converge pointwise to the discontinuous function

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 0 \end{cases}$$

even though each of them are continuous (continuous functions converge uniformly to another continuous function). In fact, $\|f_n - f\| = 1$ for all n. So the situation in C_b is more subtle.

Theorem. *Let (f_n) be a bounded sequence of continuous functions in C^0. If (f_n) is* ***equicontinuous****, then there exists a subsequence (f_{n_k}) and $f \in C^0$ such that (f_{n_k}) converges uniformly to f.*

We haven't defined equicontinuity yet, so the theorem doesn't say much. We will work up to the definition.

Lemma. *Suppose the sequence (f_n) in C_b is bounded as a sequence. Then there exists a subsequence (g_k) such that (g_k) converges pointwise on $\mathbb{Q} \cap [a, b]$.*

Examples

- Suppose (f_n) is the sequence defined by

$$f_n\left(\frac{1}{k}\right) = \frac{n \bmod k}{k} \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\right\}$$

and let $f(x) = 0$ for other x. Then, for example, we have

$$\begin{cases} f_n(1) = (0, 0, 0, \ldots) \\ f_n\left(\frac{1}{2}\right) = \left(\frac{1}{2}, 0, \frac{1}{2}, 0, \ldots\right) \\ f_n\left(\frac{1}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, 0, \frac{1}{3}, \frac{2}{3}, 0, \ldots\right) \end{cases}$$

The first functions converge to 0. We can take subsequences of these functions, examining each point individually to select a subsequence that converges on that point. See the below proof for details.

Proof. Let $Q = \{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q} \cap [a, b]$. We will build a nested family of subsequences of (f_n). Note that $f_n(q_1)$ is a bounded sequence in $\mathbb{R}$, so by Bolzano-Weierstrass there exists a subsequence (f_{n_k}) such that $(f_{n_k}(q_1))$ converges to y_1.

Look at this sequence (f_{n_k}). $f_{n_k}(q_2)$ is a bounded sequence in $\mathbb{R}$, so there is a subsequence $(f_{n_{k_\ell}}(q_2))$ that converges to y_2.

We can similarly continue in this way, defining the kth subsequence (f_n^k) of (f_n^{k-1}). Then these sequences have the property that $f_n^k(q_i)$ converges to y_i whenever $i \leq k$. So the kth subsequence converges on the first k points.

Define $g_k = f_k^k$ (the kth term of the kth subsequence). Then by construction, (g_k) is a subsequence of (f_n) such that $(g_k(q_i))$ converges to y_i for all i. $\square$

Suppose (f_n) is a sequence of continuous functions. The subsequence (g_k) from above also consists of continuous functions. We would like to show that (g_k) converges uniformly. However, since C^0 is complete, it's enough to show that (g_k) is a Cauchy sequence.

So we want to show $|g_k(x) - g_\ell(x)|$ is small for all x. We have

$$|g_k(x) - g_\ell(x)| \leq |g_k(x) - g_k(q_i)| + |g_k(q_i) - q_\ell(q_i)| + |q_\ell(q_i) - g_\ell(x)|$$

using the triangle inequality twice. The first and last terms are small by the continuity of g_k and g_ℓ, so by chooseing q_i sufficiently close to x we can bound these. The middle term is small because $(g_k(q_i))$ converges to y_i. The problem is that we don't know how these terms interact with each other.

Definition. *A sequence of functions (f_n) is **equicontinuous** if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n.*

Examples

- Define $f_n(x) = x^n$ on $[0, 1]$. The sequence (f_n) is not equicontinuous. Each f_n is continuous at $x = 1$. If $|1 - y| < \epsilon/n = \delta_n$, then $|1 - y^n| < \epsilon$. However, δ_n converges to 0 as n goes to infinity, so we cannot choose some δ such that works for all of the f_n functions.

Next time we will see some more examples of equicontinuity and then prove the Arzela-Ascoli theorem.

3/13/2019 - Arzela-Ascoli theorem, ODE existence theorem

Equicontinuity and Arzela-Ascoli

Recall that $C_b = C_b([a,b],\mathbb{R})$ is the space of bounded functions on $[a,b]$. A sequence of functions (f_n) is bounded if there exists $M > 0$ such that $\|f_n\| \leq M$ for all n (each f_n as a function is bounded by definition, but this is saying that the *norms* of these functions are also bounded by some M).

Recall that (f_n) is equicontinuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n. When we restrict our attention to one particular n, this means that each f_n is uniformly continuous. The added power of equicontinuity says that δ doesn't actually depend on the particular function f_n we are considering.

Examples

- Suppose each f_n is differentiable and $\|f_n'\| \leq M$ for all n. Then (f_n) is equicontinuous.

  Proof. We have
  $$|f_n(x) - f_n(y)| = |f'(c)(x-y)| < M \cdot |x-y|$$
  by the mean value theorem. The right hand side does not depend on n, so take $\delta < \epsilon/M$. □

- Suppose (g_n) is a bounded sequence with $\|g_n\| \leq M$ for all n. Define
  $$\phi_n(x) = \int_a^x g_n(t)\,dt$$
  Then (ϕ_n) is equicontinuous.

  Proof. We have
  $$\begin{aligned} |\phi_n(x) - \phi_n(y)| &= \left| \int_a^x g_n - \int_a^y g_n \right| \\ &= \left| \int_y^x f_n \right| \\ &\leq \|g_n\| \cdot |x-y| \\ &\leq M \cdot |x-y| \end{aligned}$$
  The right hand side does not depend on n, so take $\delta < \epsilon/M$.

  The sequence (f_n) defined by $f_n(x) = x^n$ on $[0,1]$ is not equicontinuous. This is reflected in the fact that $f_n'(1) = n$, so the norms of the derivatives f_n' are unbounded. □

Theorem (Arzela-Ascoli Theorem)**.** *Suppose (f_n) is a sequence of bounded functions that is itself bounded and equicontinuous. Then there exists a subsequence (g_k) of (f_n) and a function $g \in C^0$ such that g_k converge to g.*

Proof. Take $Q = \mathbb{Q} \cap [a, b]$, and enumerate these by $Q = \{q_1, q_2, \ldots\}$. Last time we showed there there exists a subsequence (g_k) such that the sequence $(g_k(q))$ in $\mathbb{R}$ converges to q for all $q \in Q$.

The main claim is that this sequence is Cauchy, namely that for all $\epsilon > 0$ there exists $N > 0$ such that $k, \ell > N$ implies $|g_k(x) - g_\ell(x)| < \epsilon$ for all x. Given the claim, there exists such a $g \in C^0$ because the metric space C^0 is complete.

Now to show the claim, fix $\epsilon > 0$. (g_k) is equicontinuous, as it's a subsequence of a sequence of equicontinuous functions. Then there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|g_k(x) - g_k(y)| < \epsilon/3$ for all k. Choose $m > 0$ such that every $x \in [a, b]$ is within δ of at least one of $q_1, q_2, \ldots, q_m$. We can do this because $Q \subset [a, b]$ is dense, so going far enough along in this sequence guarantees that we are within δ of at least one of these.

Next, choose $N > 0$ such that $k, \ell > N$ implies $|g_k(q_i) - g_\ell(q_i)| < \epsilon/3$ for all $1 \leq i \leq m$. We can do this because each sequence $(g_k(q_i))$ in $\mathbb{R}$ converges (and is Cauchy), so let N be the maximum of the N_i obtained from each of these sequence.

Then if $k, \ell > N$, for each x choose q_i such that $|x - q_i| < \delta$. This yields

$$\begin{aligned} |g_k(x) - g_\ell(x)| &\leq |g_k(x) - g_k(q_i)| + |g_k(q_i) - g_\ell(q_i)| + |g_\ell(q_i) - g_\ell(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which proves the claim, completing the proof. $\square$

Exercise

Consider the sequences of functions (f_n) and (g_n) defined by

$$\begin{aligned} f_n(x) &= \sin(2\pi n x) \\ g_n(x) &= n \cdot \sin(\frac{\pi x}{n}) \end{aligned}$$

One of these is bounded/equicontinuous, and the other is not.

(g_n) is equicontinuous. Taking the derivative yields

$$|g_n'(x)| = |\pi \cos(\frac{\pi x}{n})| \leq \pi$$

So the derivatives are bounded. By the example above, this implies (g_n) is equicontinuous.

(f_n) is not equicontinuous. The functions oscillate more rapidly, and their derivatives are increasing without bound.

Then by the theorem, (g_n) has a subsequence (g_{n_k}) that converges to some g. To determine g, note that $\sin(\pi x/n)$ has Taylor polynomial

$$P(x) = \frac{\pi x}{n} - (\frac{\pi}{n})^3 \frac{x^3}{3!} + (\frac{\pi}{n})^5 \frac{x^5}{5!} + \dots$$

Multiplying $P(x)$ by n allows us to guess that g_n converge to g, where $g(x) = \pi x$. This is in fact true.

Peano's existence for solutions to ODEs

Theorem (Peano's Theorem (1886))**.** *Consider an intial value problem*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

If the function f is continuous near (t_0, x_0), then this problem has a solution near t_0. Namely, there exists $\phi : (t_0 - \alpha, t_0 + \alpha) \to \mathbb{R}$ such that $\phi(t_0) = x_0$ and $\phi'(t) = f(t, \phi(t))$.

Euler's approximation method

- Choose a step size $h > 0$.
- Start at (t_0, x_0). Then move along the line of slope $f(t, x_0)$ for time h.
- Set $(t_1, x_1) = (t_0 + h, x_0 + f(t_0, h_0)h)$.
- Follow the line of slope $f(t_1, x_1)$ for time h, and continue to repeat this process.

This method yields a numerical approximation for a solution as h becomes small.

To find an actual solution, the idea will be to take a sequence ϕ_k of approximations with step size h_k, which goes to 0. Then we will show ϕ_k has a convergent subsequence using Arzela-Ascoli and show that the limit is indeed a solution. We will sketch a proof with four steps.

Proof. Step 1. Translate the problem into something more tractable using the fundamental theorem of calculus. If we have a solution ϕ such that $\phi'(t) = f(t, \phi(t))$ for all t is some interval, then integrating yields

$$\int_{t_0}^{t} \phi'(s)\, ds = \int_{t_0}^{t} f(s, \phi(s))\, ds$$

By the fundamental theorem of calculus this is

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds \tag{1}$$

Conversely, given this equality, differentiation yields

$$\phi'(t) = f(t, \phi(t))$$

So it suffices to find a function that satisfies (1).

Step 2. Use Euler approximations ϕ_k with step size $h(k)$ that goes to 0. Some care should be taken when choosing these step sizes, but we won't go into this now.

Each ϕ_k is piecewise linear, so we can write

$$\phi_k(t) = x_0 + \int_{t_0}^{t} \phi_k'(s)\, ds$$

This follows from the fundamental theorem of calculus. The only issue is that ϕ_k is not differentiable at the transition points, but this is not a problem, as we can just break up the integral for

$$\begin{aligned}\phi_k(t) &= x_0 + (\phi_k(t_1) - \phi_k(t_0)) + (\phi_k(t) - \phi_k(t_1)) \\ &= x_0 + \int_{t_0}^{t_1} \phi_k'(s)\, ds + \int_{t_1}^{t} \phi_k'(s)\, ds\end{aligned}$$

again using the fundamental theorem of calculus.

Step 3. The sequence (ϕ_k) is bounded and equicontinuous. Define

$$\Delta_k(t) = \begin{cases} \phi_k'(t) - f(t, \phi_k(t)) & t \neq t_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

The function Δ_k measures the difference betwen the slope of our piecewise linear approximation and the actual slope of the lines given by the differential equation at a point.

This yields

$$\begin{aligned}\phi_k(t) &= x_0 + \int_{t_0}^{t} \phi_k'(s)\, ds \\ &= x_0 + \int_{t_0}^{t} f(s, \phi_k(s)) + \Delta_k(s)\, ds\end{aligned}$$

We want to show that the integrand is bounded independent of k. This implies that the derivatives ϕ_k' are bounded, which means (ϕ_k) is bounded and equicontinuous. f is continuous near (t_0, x_0), so it is bounded near (t_0, x_0). We also know

$$|\Delta_k(t)| = |f(t_i, x_i) - f(t, \phi(t))|$$

We can ensure Δ_k is small by taking a small step size, which ensures that these two terms are not too far apart (as f is uniformly continuous, since it is continuous near (t_0, x_0)). In fact, (Δ_k) will converge uniformly to 0 near (t_0, x_0). Since both of the summands in the integrand above are bounded, (ϕ_k) is bounded and continuous.

Step 4. Apply Arzela-Ascoli to obtain a subsequence of (ϕ_k) that converges to ϕ uniformly. Replace

ϕ_k with this subsequence. We must show that the limit ϕ satisfies equation (1). We have

$$\begin{aligned}
\phi(t) &= \lim_{k\to\infty} \big(x_0 + \phi_k(t)\big) \\
&= x_0 + \lim_{k\to\infty} \int_{t_0}^{t} f(s, \phi_k(s)) + \Delta_k(s)\, ds \\
&= x_0 + \int_{t_0}^{t} \lim_{k\to\infty} \Big(f(s, \phi_k(s)) + \Delta_k(s) \Big)\, ds \\
&= x_0 + \int_{t_0}^{t} f(s, \varphi(s)\, ds
\end{aligned}$$

as the Δ_k converge to 0. $\square$

3/25/2019 - Multivariable derivative, chain rule

Today we will start developing calculus in higher dimensions, starting with the derivative.

Multivariable derivatives

Definition. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is* ***differentiable*** *at $a \in \mathbb{R}^n$ if there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - T(h)}{|h|} = 0$$

Note that h is a vector in $\mathbb{R}^n$, so we must take the norm in the denominator for this limit to make sense, as we cannot divide by a vector. The numerator is a vector in $\mathbb{R}^m$, so the limit is taken as a vector (namely the limit of this expression is the zero vector in $\mathbb{R}^m$).

Examples

- In the one-dimensional case, the multivariable derivative and the usual derivative coincide. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable in the usual sense at $a \in \mathbb{R}$, we can take our linear map to be $T(h) = f'(a) \cdot h$. Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+h) - f(a) - T(h)}{|h|} &= \lim_{h\to 0} \frac{|f(a+h) - f(a) - f'(a)\cdot h|}{|h|} \\ &= \lim_{h\to 0} \left| \frac{f(a+h) - f(a)}{|h|} - f'(a) \right|\end{aligned}$$

  So f is also differentiable in the multivariable sense.

Proposition. *If f is differentiable at $a \in \mathbb{R}^n$, the linear map T is unique.*

Proof. The proof is left as an exercise. □

Remark. *If f is differentiable at a with associated linear map T, denote $Df(a) = T$.*

Examples

- If $f(x) = c$ for any $x \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$, you can check that $Df(a) = 0$ for all $a \in \mathbb{R}^n$.
- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $DT(a) = T$ for all $a \in \mathbb{R}^n$.

The derivative at a point is a *linear map* between the domain and codomain of the function f. It is the unique linear map that best approximates the function at this point. In this context, the single-variable derivative $f'(a)$ should be understood as a linear map from $\mathbb{R}$ to $\mathbb{R}$ given by multiplication by $f'(a)$. You can view this linear map abstractly, or alternatively choose a basis to represent it as a matrix. We'll discuss this idea further later.

Examples

- If $f : \mathbb{R} \to \mathbb{R}^2$ is a differentiable curve in the plane, we can view f as the trajectory of a particle in $\mathbb{R}^2$. Then the derivative $Df(a)$ is a matrix in $M_{2\times 1}(\mathbb{R})$ which is the velocity vector.
- If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then $Df(a) \in M_{1\times 2}(\mathbb{R})$. The gradiant $f(a)$ is defined as $Df(a)^t \in M_{2\times 1}(\mathbb{R})$. This vector points in the direction here f is increasing fastest at a. For example, if we take $f(x, y) = x^2 + y^2$, you can check

  $$Df(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

  Then $f(a) = Df(a)^t$ points radially, in the direction in which f is increasing fastest.

These examples should give a geometric meaning about the derivative.

Definition. *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$, then for any $v \in \mathbb{R}^n$, the **directional derivative** at a is given by*

$$D_v f(a) = \lim_{t\to 0} \frac{f(a + tv) - f(a)}{t}$$

This limit exists, and it is equal to $Df(a)(v)$.

The directional derivative captures the rate of change of f as we move in the direction given by v. The claim is that total differentiability of f implies that the directional derivatives exist and are equal to $Df(a)(v)$.

Proof. Since f is differentiable at a, we can define

$$R(h) = f(a + h) - f(a) - Df(a)(h)$$

R is the numerator of the limit in the definition of the derivative, so by assumption

$$\lim_{h\to 0} \frac{R(h)}{|h|} = 0$$

We can then compute the quantity

$$\begin{aligned} \frac{f(a + tv) - f(a)}{t} &= \frac{Df(a)(tv) + R(tv)}{t} \\ &= \frac{tDf(a)(v) + R(tv)}{t} \\ &= Df(a)(v) + \frac{R(tv)}{t} \end{aligned}$$

by the linearity of $Df(a)$. We can rewrite the second term as

$$\frac{R(tv)}{|tv|} \cdot \frac{|tv|}{t}$$

By assumption, the first term goes to zero as t goes to zero. The second term is also $\pm|v|$, so the entire expression goes to zero as t goes to zero. This implies that the limit exists and

$$\begin{aligned}\lim_{t\to 0}\frac{f(a+tv)-f(a)}{t} &= \lim_{t\to 0} Df(a)(v) + \frac{R(tv)}{|tv|}\cdot\frac{|tv|}{t}\\ &= Df(a)(v)\end{aligned}$$

as desired. □

Remark. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable at $a\in\mathbb{R}^n$. When $v=e_i$ is a standard basis vector, $Df(a)(e_i)$ is the ith* ***partial derivative*** *of f, denoted $D_i f(a)$. By the above proof, the matrix for $Df(a)$ with respect to the standard basis is then given by*

$$Df(a) = \begin{pmatrix} D_1 f(a) & D_2 f(a) & \dots & D_n f(a)\end{pmatrix}$$

Remark. *If f is differentiable at a, then all of the directional derivatives $D_v f(a)$ exist for all $v\in\mathbb{R}^n$. However, the converse is false. Even if the directional derivatives exist for all v, it is not guaranteed that f is differentiable.*

Examples

- Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & (x,y)\neq 0\\ 0 & (x,y)=0\end{cases}$$

However, it is possible to remedy this issue. We'll prove the following result next time.

Theorem (Continuous Partials theorem)**.** *Let $A\subset\mathbb{R}^n$ be open and $f:A\to\mathbb{R}$ be a function. If $D_i f(x)$ exists for all i and each $D_i f(x)$ is continuous as a function from A to $\mathbb{R}$ (given by taking $x\in A$ to $D_i f(x)$), then f is differentiable at all points in A.*

Note that we take $A\subset\mathbb{R}^n$ to be open. We can only make sense of differentiation for functions defined on open subsets of $\mathbb{R}^n$. This is because to take the limit as h approaches 0, we are evaluating f on some open ball around a. To do this, we need to know that the domain of f is open.

Lemma. *If $f:\mathbb{R}^n\to\mathbb{R}^m$ is differentiable, then f is continuous.*

Proof. To show that f is continuous, we want that

$$\lim_{h\to 0}|f(a+h)-f(a)|=0$$

Using the definition of differentiability and the function R as defined above, we have that

$$\begin{aligned}|f(a+h)-f(a)| &= |Df(a)(h)+R(h)|\\ &\le |Df(a)(h)|+|R(h)|\end{aligned}$$

As h goes to zero, the first term will go to zero because $Df(a)$ is a linear map (and hence continuous). By assumption, we know that $R(h)/|h|$ goes to zero, so $R(h)$ individually certainly must go to zero. This proves that $|f(a+h)-f(a)|$ goes to zero, so f is continuous. □

Proposition. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are differentiable and $c \in \mathbb{R}$ is a constant, then*

1. *$D(f + cg) = D(f) + cD(g)$. Namely, the action of taking the derivative itself is linear (this is distinct from the idea that $Df(a)$ is a linear map).*
2. *$D(g \circ f) = Dg \circ Df$. This is the chain rule.*
3. *$D(fg) = (Df) \cdot g + f \cdot (Dg)$. This is the Leibniz product rule.*

Proving linearity is easy, and the product rule is on the homework. We'll prove the chain rule.

Proof. We have the diagram

$$\mathbb{R}^n \xrightarrow{\quad f \quad} \mathbb{R}^m \xrightarrow{\quad g \quad} \mathbb{R}^p$$

$$a \longrightarrow b \longrightarrow c$$

We want to show that $g \circ f$ is differentiable and $Dg(b) \circ Df(a) = D(g \circ f)(a)$. Let $b = f(a)$, $A = Df(a)$, $c = g(b)$, and $B = Dg(b)$. Then

$$\begin{aligned} f(a+h) &= b + A(h) + R_f(h) \\ g(b+k) &= c + B(k) + R_g(k) \end{aligned}$$

Consider the expression

$$R_{g\circ f}(h) = g(f(a+h)) - g(f(a)) - B(A(h))$$

If we can show that

$$\lim_{h\to 0} \frac{R_{g\circ f}(h)}{|h|} = 0$$

this would imply that the map $g \circ f$ is differentiable and the map BA is indeed the derivative of $g \circ f$. Plug in our above expression for $f(a+h)$ to yield

$$\begin{aligned} R_{g\circ f}(h) &= g(b + A(h) + R_f(h)) - g(f(a)) - BA(h) \\ &= g(b + k(h)) - g(f(a)) - BA(h) \\ &= c + B(A(h) + R_f(h)) + R_g(k(h)) - c - BA(h) \\ &= BR_f(h) + R_g(k(h)) \end{aligned}$$

where we are defining $k(h) = A(h) + R_f(h)$. We also used the linearity of B to cancel terms in the third line. If we can show that

$$\lim_{h\to 0} \frac{|BR_f(h)|}{|h|} = 0 \tag{1}$$

$$\lim_{h\to 0} \frac{|R_g(k(h))|}{|h|} = 0 \tag{2}$$

then we will have proven the claim. Since B is a linear map, we can bound it for

$$|BR_f(h)| \le M_B |R_f(h)|$$

We know that $R_f(h)/|h|$ goes to zero as h goes to zero, since f is differentiable. This implies that $R_f(h)$ itself goes to zero, so the product $|BR_f(h)|$ goes to zero, which shows the first desired limit statement (1).

For the second limit statement, there are two cases. If $k(h) = 0$, then $R_g(k(h)) = 0$. Otherwise, we have

$$\frac{R_g(k(h))}{|h|} = \frac{R_g(k(h))}{|k(h)|} \cdot \frac{|k(h)|}{|h|}$$

Recall $k(h) = A(h) + R_f(h)$, so as h goes to zero k also goes to zero (these are all continuous functions). So we can apply the differentiability of g to conclude that the first term in the product goes to zero. We can bound the second term in the product by

$$\frac{|k(h)|}{|h|} \leq \frac{|A(h)|}{|h|} + \frac{|R_f(h)|}{|h|}$$

We can bound $|A(h)| \leq M_A|h|$, so $|A(h)|/|h|$ goes to zero, while $|R_f(h)|/|h|$ goes to zero because f is differentiable. Hence the second term from above is bounded, so we have proven the second limit statement (2). This completes the proof. $\square$

Examples

- Let $A, B \subset \mathbb{R}^n$ be open sets and $f : A \to B$ and $g : B \to A$ be differentiable functions. If $g \circ f = id_A$, then we can compute the derivative for $x \in A$ by

$$I = D(id_A)(x) = D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$$

  Hence we have

$$Dg(f(x))^{-1} = Df(x)$$

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x - 7y + 3xy^2$$

  Then compute the partial derivatives, taking one variable to be constant and differentiating with respect to the other.

$$\begin{aligned} D_1 f(x, y) &= 1 + 3y^2 \\ D_2 f(x, y) &= -7 + 6xy \end{aligned}$$

  Then by the continuous partials theorem we have

$$\begin{pmatrix} 1 + 3y^2 & -7 + 6xy \end{pmatrix}$$

  If we want to know how f changes in the direction $(1, 2)$ at the point $(-2, -1)$, we should compute

$$Df(-2, -1) \cdot (1, 2) = (4, 5) \cdot (1, 2) = 14$$

  The dot product notation is the same as multiplying the matrix for $Df(-2, -1)$ given by $(4, 5)$ with the column vector $(4, 5)$.

3/27/2019 - Continuous partials theorem, multivariable MVT, higher derivatives

Recall that say a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if there exists a linear map (the derivative) $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Df(a)(h)}{|h|} = 0$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$, then we have

$$Df(a) = \begin{pmatrix} D_1 f(a) & D_2 f(a) & \dots & D_n f(a) \end{pmatrix}$$

where

$$D_i f(a) = \lim_{t\to 0} \frac{f(a+te_i) - f(a)}{t}$$

is the ith partial derivative.

Geometrically, we should understand the derivative as the best linear approximation to f at a point. For example, in the one-dimensional case, the function

$$P_1(x) = f(a) + f'(a)(x-a)$$

is the tangent line best approximating f at a. In the higher dimensional case, the hyperplane

$$P_1(x) = f(a) + Df(a)(x-a)$$

is the tangent plane best approximating f at a.

Continuous partials theorem

Today we will prove the following result.

Theorem (Continuous Partials theorem). *Let $A \subset \mathbb{R}^n$ be open. Let $f : A \to \mathbb{R}$ be a function, and suppose $D_i f(a)$ exists for all $a \in A$ and all i. Further suppose that the functions*

$$\begin{aligned} D_i f : A &\to \mathbb{R} \\ a &\mapsto D_i f(a) \end{aligned}$$

are continuous for all i. Then f is differentiable on all of A.

Examples

- Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = x - 7y + 3xy^2$$

The partial derivatives

$$\begin{aligned} D_1 f(x,y) &= 1 + 3y^2 \\ D_2 f(x,y) &= -7 + 6xy \end{aligned}$$

exist and are continuous. By the continuous partials theorem, f is differentiable and

$$Df(a) = \begin{pmatrix} D_1 f(a) & D_2 f(a) \end{pmatrix}$$

- To see why the continuity of the partial derivatives is an essential assumption, consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

To compute the directional derivative at $(0, 0)$ in direction (v_1, v_2), observe that we have

$$\begin{aligned} D_v g(0, 0) &= \lim_{t \to 0} \frac{g(tv) - g(0)}{t} \\ &= \lim_{t \to 0} \frac{\frac{t^3 v_1^2 v_2}{t^4 v_1^4 - t^2 v_2^2}}{t} \\ &= \lim_{t \to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} \\ &= \begin{cases} \frac{v_1^2}{v_2} & v_2 \neq 0 \\ 0 & v_2 = 0 \end{cases} \end{aligned}$$

However, g is not differentiable at $(0, 0)$. For if g were differentiable at $(0, 0)$, then we would have

$$Dg(0, 0) = \begin{pmatrix} D_1 g(0, 0) & D_2 g(0, 0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

This implies

$$D_v g(0, 0) = Dg(0, 0)(v) = 0$$

which contradicts the above computation of the directional derivatives.

The upshot is that if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then

$$\begin{aligned} Df(a, b) : \mathbb{R}^2 &\to \mathbb{R} \\ v &\mapsto Df(a, b)(v) \end{aligned}$$

should be a linear map. In the first example, we have

$$Df(a, b) : (v_1, v_2) \mapsto (1 + 3b^2)v_1 + (-7 + 6ab)v_2$$

which is linear. But in the second example, we have

$$Dg(a, b) : (v_1, v_2) \mapsto \begin{cases} \frac{v_1^2}{v_2} & v_2 \neq 0 \\ 0 & v_2 = 0 \end{cases}$$

which is not linear.

Multivariable mean value theorem

To prove the continuous partials theorem, we will need a multivariable generalizationo of the mean value theorem.

Theorem (Multivariable Mean Value theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $a, a+h \in \mathbb{R}^n$. Then there exists a point $c = a + sh$, with $s \in (0,1)$, such that*

$$f(a+h) - f(a) = Df(c)(h)$$

The idea of the proof will be to reduce to the one-dimensional case.

Proof. Consider the function

$$g(t) = a + th$$

Then define $H = f \circ g : [0,1] \to \mathbb{R}$. Since f and g are differentiable and H is a function from $\mathbb{R}$ to $\mathbb{R}$, we can apply the one-dimensional mean value theorem to H. So there exists $s \in (0,1)$ such that

$$H(1) - H(0) = H'(s) \cdot 1$$

The left side is

$$f(g(1)) - f(g(0)) = f(a+h) - f(a)$$

By the chain rule, the right side is

$$(f \circ g)'(s) = Df(g(s)) \circ Dg(s)$$

Take $c = g(s) = a + sh$. $Dg(s)$ is just h, so this yields

$$f(a+h) - f(a) = Df(c)(h)$$

□

Remark

- The multivariable mean value theorem does not hold if we switch the domain and codomain. For example, consider the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by

  $$f(t) = (\cos t, \sin t)$$

  f traces out a circle in the plane. On the interval $[0, \pi]$, we are looking for $c \in (0, \pi)$ such that

  $$(-2, 0) = f(\pi) - f(0) = f'(c)(\pi - 0) = \pi(-\sin c, \cos c)$$

  But this is impossible, as the norm of the left side is 2 while the norm of the right side is always π.

Continuous partials theorem

We can now prove the continuous partials theorem.

Proof. Fix $a \in A$. Then examine the matrix

$$T = \begin{pmatrix} D_1 f(a) & \dots & D_n f(a) \end{pmatrix}$$

To show f is differentiable, we want to prove

$$\lim_{t\to 0} \frac{f(a+h)-f(a)-T(h)}{|h|} = 0$$

For concreteness, we will work when $n=3$, but the generalized proof is identitical. Define

$$R(h) = f(a+h) - f(a) - T(h)$$

Write[5]

$$h = h_1 e_1 + h_2 e_2 + h_3 e_3$$

$$\begin{aligned} p_0 &= a \\ p_1 &= p_0 + h_1 e_1 \\ p_2 &= p_1 + h_2 e_2 \\ p_3 &= p_2 + h_3 e_3 \end{aligned}$$

Then we have

$$f(a+h) - f(a) = f(p_3) - f(p_0) = \sum_{i=1}^{3} f(p_i) - f(p_{i-1})$$

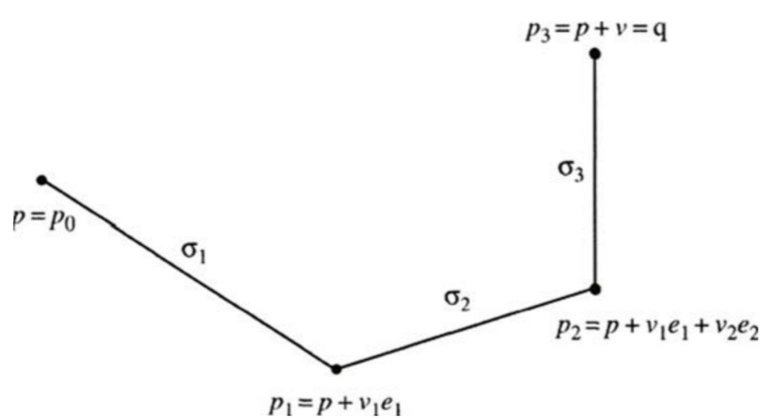
Hence

$$\begin{aligned} R(h) &= f(a+h) - f(a) - T(h) \\ &= \left(\sum_{i=1}^{3} f(p_i) - f(p_{i-1})\right) - \begin{pmatrix} D_1 f(a) & D_2 f(a) & D_3 f(a) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \\ &= \sum_{i=1}^{3} \Big(f(p_i) - f(p_{i-1}) - D_i f(a) h_i \Big) \end{aligned}$$

Since $p_i = p_{i-1} + h_i e_i$, by the single variable mean value theorem applied to f composed with the map $t \mapsto p_{i-1} + t_i e_i$ there exists $t_i \in (0, h_i)$ such that

$$f(p_i) - f(p_{i-1}) = Df(p_{i-1} + t_i e_i)(h_i e_i)$$

[5] We have the following diagram, where v is used in place of h and p in place of a.



$h_i e_i$ is the vector with h_i in the ith coordinate, so multiplication picks out the ith coordinate of $Df(p_{i-1} + t_i e_i)$ for

$$f(p_i) - f(p_{i-1}) = D_i f(p_{i-1} + t_i e_i) h_i$$

Let the quantity $p_{i-1} + t_i e_i$ be labelled q_i. Replacing this equality in the above expression for $R(h)$ yields

$$R(h) = \sum_{i=1}^{3} \Big(D_i f(q_i) - D_i f(a) \Big) h_i$$

$$\frac{R(h)}{|h|} = \sum_{i=1}^{3} \underbrace{\Big(D_i f(q_i) - D_i f(a) \Big)}_{\text{continuity}} \underbrace{\frac{h_i}{|h|}}_{\leq 1}$$

But as h goes to zero, the continuity of the partials implies that the first term goes to zero, while we know that the second term is always bounded by 1. This shows that

$$\lim_{h \to 0} \frac{R(h)}{|h|} = 0$$

So f is differentiable at a. □

Definition. *A function f with continuous partials is called **continuously differentiable** or C^1.*

We have shown the inclusions

$$\{C^1 \text{ functions}\} \subsetneq \{\text{differentiable functions}\} \subsetneq \{\text{functions with partial derivatives}\}$$

The strictness of the first inclusion is on the homework, while the strictness of the second inclusion was demonstrated earlier today with the function g.

Higher derivatives

Te function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x \sin y + e^{xy}$$

has partial derivatives

$$\begin{aligned} D_1 f(x, y) &= \sin y + y e^{xy} \\ D_2 f(x, y) &= x \cos y + x e^{xy} \end{aligned}$$

However, the partial derivatives themselves are again differentiable, with partial derivatives

$$\begin{aligned} D_1 D_1 f(x, y) &= y^2 e^{xy} \\ D_2 D_1 f(x, y) &= \cos y + e^{xy} + xy e^{xy} \\ D_1 D_2 f(x, y) &= \cos y + e^{xy} + xy e^{xy} \\ D_2 D_2 f(x, y) &= -x \sin y + x^2 e^{xy} \end{aligned}$$

The derivatives of the partials of f are again differentiable. In such a case, f is a C^2 function. Observe that $D_1 D_2 f$ and $D_2 D_1 f$ are the same. This holds in general.

Theorem (Clairaut's theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2, then $D_iD_jf(p) = D_jD_if(p)$ for all $p \in \mathbb{R}^n$ and for all i, j.*

You will prove this on the homework later with Fubini's and Stokes' theorem. We can restate this by introducing the notion of a *Hessian*.

Definition. *For a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$, the **Hessian matrix** for f is the $n \times n$ matrix $H(f) = (D_jD_if)$.*

Theorem (Clairaut's theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2, then $H(f)$ is symmetric.*

Just like a single-variable functions, a function $f : \mathbb{R}^2 \to \mathbb{R}$ has Taylor polynomials defined at $(a, b) \in \mathbb{R}^2$. For $p = (0, 0)$, the 2nd order approximation for f at p is

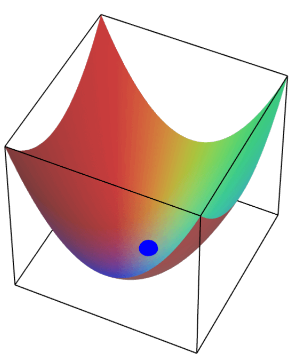
$$P_2(x, y) = f(0) + (D_1f(0)x + D_2f(0)y) + \frac{1}{2}\left(D_1^2f(0)x^2 + 2D_1D_2f(0)xy + D_2^2f(0)y^2\right)$$

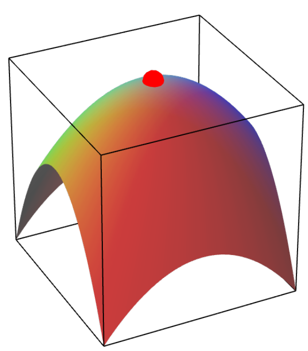
There is similarly an analogue for the 2nd derivative test. If $Df(0) = 0$, then use the fact that $H(f)$ is symmetric. Since $H(f)$ is symmetric, it is diagonalizable. Then up to a change of basis, the 2nd order Taylor polynomial can be written

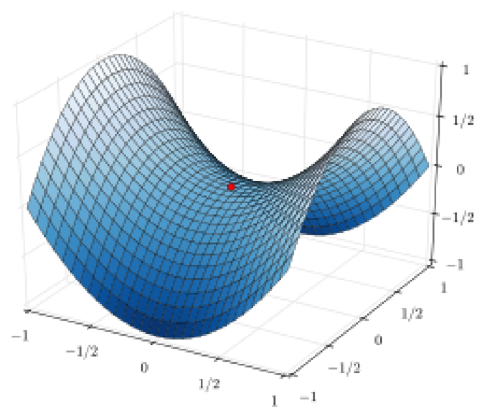
$$P_2(u, v) = f(0) + Au^2 + Bv^2$$

where A and B are constants that depend on the second order partials of f (as the non-diagonal terms will disappear with the appropriate basis change).

If A and B are both positive, then locally f is an upwards facing surface, in which case $(0, 0)$ is a local minimum. If A and B are both negative, then locally f is a downwards facing surface, in which case $(0, 0)$ is a local maximum. If the signs of A and B differ, then $(0, 0)$ is a saddle point.







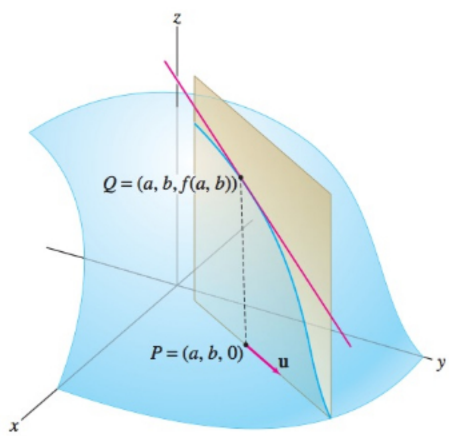
4/1/2019 - Implicit function theorem

Today we will discuss three applications of the multivariable derivative. Before that, let's review some important concepts.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Then the derivative of f at p is a linear map $Df(p) : \mathbb{R}^n \to \mathbb{R}$. We can interpret

$$Df(p)(v) = D_v f(p)$$

as the directional derivative of f in the direction v. It measures how the function f changes along the line v in the domain.



We can also write

$$\begin{aligned} Df(p)(v) &= \begin{pmatrix} D_1 f(p) & \dots & D_n f(p) \end{pmatrix} \begin{pmatrix} v_1 \\ \dots \\ v_n \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} D_1 f(p) \\ \dots \\ D_n f(p) \end{pmatrix}}_{f(p)} \cdot \begin{pmatrix} v_1 \\ \dots \\ v_n \end{pmatrix} \end{aligned}$$

where $\nabla f(p)$ is the ***gradient*** of f and $\cdot$ is the dot product.

Question: For which v with $|v| = 1$ is $D_v f(p)$ the largest? The smallest?

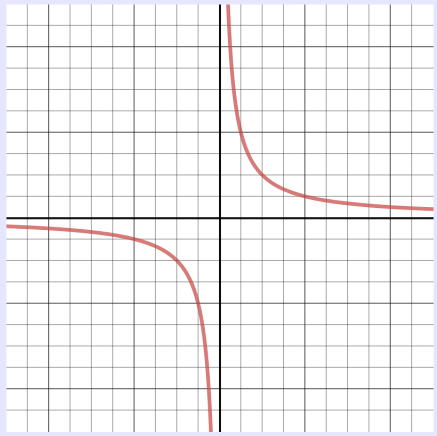
Answer: By examining the above expression for $D_v f(p) = Df(p)(v)$, we can see that the dot product is maximized when v is parallel to $\nabla f(p)$. It is minimized when v is perpendicular to $\nabla f(p)$.

So $\nabla f(p)$ is the vector that represents the direction in which f is changing the fastest.

The above remark should illustrate that *the gradient is orthogonal to the level sets of f*. We can see this with an example.

Examples

- Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$. If we consider the level set given by $f(x, y) = 1$, then we have the graph



The gradient vector at each point is perpendicular to the graph, as $Df(x, y) = \begin{pmatrix} y & x \end{pmatrix}$.

Implicit function theorem

The implicit function theorem is about solving *nonlinear equations*.

Examples

- If we have the equation
$$x^2 - 5xy + y^3 = 8$$
we would like to separate values, namely by writing y in terms of x or x in terms of y. However, nonlinearity makes this difficult. We can find some solutions explicitly (for example if $x = 0$ then $y^3 = 8$, so $(0, 2)$ is a solution).

We can then ask if there is a solution near a point we know to be a solution. The approach will be to view the set of solutions as the level set of some function.

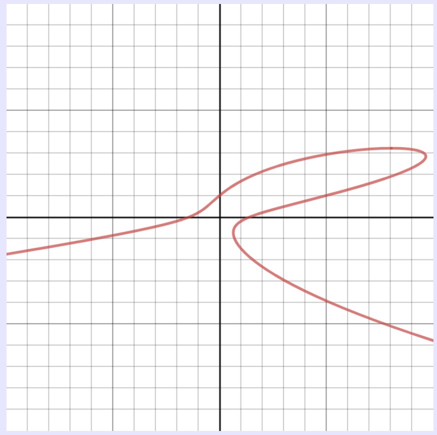
Definition. *Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$, the **level set** of f is*
$$\chi_c = \{(x, y) : f(x, y) = c\}$$
The level set consists of solutions to the equation $f(x, y) = c$.

Suppose χ_c is the graph of a function, namely that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that
$$\chi_c = \text{graph}(g) = \{(x, g(x)) : x \in \mathbb{R}\}$$
Then we can find all of the solutions. For each x, we know $f(x, g(x)) = c$, and furthermore all solutions have this form.

If we can write the level set as the graph of a function, then we know how to understand the solutions. However, in general this is too much to ask for.

Examples

- The 8−level set of the function $x^2 - 5xy + y^3$ is



This is not given as the graph of any function.

However, the key idea is that at certain points the level set may look like the graph of a function *locally* (for example, when x is negative in the above example). The problem occurs when the gradient vector is parallel to the x-axis, namely when $D_2 f(p) = 0$.

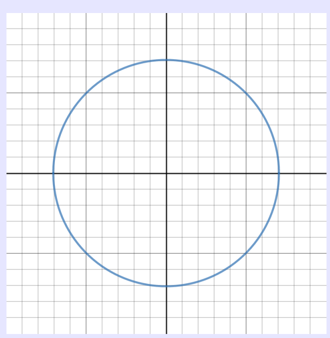
Theorem (Implicit Function theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Let $p \in \mathbb{R}^n$ and $c = f(p)$. If $D_n f(p) \neq 0$, then there exists a neighborhood $U \times V \subset \mathbb{R}^{n-1} \times \mathbb{R}$ of p and a unique function $g : U \to V$ so that*

$$\chi_c \cap (U \times V) = \textit{graph}(g)$$

Moreover, g is C^1.

Examples

- The function $f(x, y) = x^2 + y^2$ has level sets given by circles.



Its derivative is

$$Df(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

Then $D_2 f(x, y) \neq 0$ if and only if $y \neq 0$. Indeed, for any point (x, y) on the circle with

$y \neq 0$, the circle is given locally as the graph of the function

$$g(x) = \sqrt{c - x^2}$$

(or the negative square root).

- Consider the function $f(x, y) = xy$. The level set at 0 is

$$\chi_0 = \{(x, y) : xy = 0\}$$

is given as the union of the x and y axes. For what points is χ_0 locally the graph of a function? Every point (x, y) with either $x \neq 0$ or $y \neq 0$ is locally the graph of a function, as the derivative of f is

$$Df(x, y) = \begin{pmatrix} y & x \end{pmatrix}$$

Note that the implicit function theorem guarantees the existence of such a function g even when we cannot find the function explicitly.

We can also consider when a level set is locally the graph of a function of y. In general, the implicit function theorem tells you when the level set is the graph of a function of the first $(n-1)$ variables. We can choose any particular variable and write the level set as a function of the remaining ones.

Inverse function theorem

When does a C^1 function $f : \mathbb{R}^n \to \mathbb{R}^m$ have a C^1 inverse? When does such a function have an inverse at least locally?

Theorem (Inverse Function theorem)**.** *Let $f : (a, b) \to \mathbb{R}$ be a C^1 function. If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is injective and the inverse of $f : (a, b) \to f((a, b))$ is also C^1.*

You will prove this one-dimensional version of the theorem on the homework. The main obstruction to invertibility is injectivity, but if $f'(x) \neq 0$ we know that f is injective.

Examples

- Consider the function $f : (-\pi, \pi) \to \mathbb{R}$ defined by $f(x) = 2x + \sin x$. Then $f'(x) = 2 + \cos x$, which is always strictly greater than 0. Then the inverse of f is a C^1 function

$$g : (-2\pi, 2\pi) \to (-\pi, \pi)$$

However, there is no easy formula for g.

- $f(x) = x^2$ has no global inverse, but it has local inverses given by $\sqrt{x}$ and $-\sqrt{x}$.

If f has an inverse g that is differentiable, then if f is C^1 we know g is C^1 as well. This follows from the chain rule. We know $y = f(g(y))$, and differentiating this yields

$$1 = f'(g(y))g'(y)$$

$$g'(y) = \frac{1}{f'(g(y))}$$

We know f' is continuous by assumption, so we have expressed g' as the composition of continuous functions.

We have an analogous result in the higher dimensional setting.

Theorem (Inverse Function theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. If $Df(p)$ is invertible, then there exists a neighborhood U of p such that $f|_U : U \to f(U)$ is bijective with a C^1 inverse.*

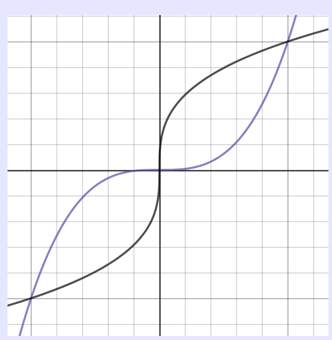
Examples

- Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (xy, y^2 - x^2)$. The inverse function theorem says that if we want to know if f has a local inverse we can examine the derivative
$$Df(x, y) = \begin{pmatrix} y & x \\ -2x & 2y \end{pmatrix}$$
$Df(x, y)$ is invertible when
$$\det(Df(x, y)) = 2y^2 + 2x^2$$
is nonzero. This is true whenever $(x, y) \neq (0, 0)$.

  Hence f is locally invertible around any point $(x, y) \neq (0, 0)$. However, note that $f(1, 1) = f(-1, -1)$, so the neighborhood U of $(1, 1)$ cannot contain the point $(-1, -1)$. We only have inverses *locally*.

  In fact, since $f(x, y) = f(-x, -y)$ we know that f is not invertible around the origin, since any neighborhood of $(0, 0)$ will contain a pair of these antipodal points. Although the inverse function theorem is not a biconditional result (an 'if and only if'), this shows that f does not have an inverse around $(0, 0)$.

- The function $f(x) = x^3$ is a bijection of $\mathbb{R}$ with inverse $g(x) = \sqrt[3]{x}$. f is C^1, but g is not. In fact, $g'(0)$ does not even exist.



  This doesn't contradict the above remark because g is not differentiable at 0. It doesn't contradict the theorem, because $f'(0) = 0$. The existence of a derivative does not imply that the derivative is differentiable.

Lagrange multipliers

We would like to find the maximum or minimum of a function[6] $\phi : \mathbb{R}^n \to \mathbb{R}$ subject to the constraint $f(x) = c$, where $f : \mathbb{R}^n \to \mathbb{R}$.

Examples

- What is the point on the parabola $y = (x-2)^2 + 2$ closest to the origin? We are looking to minimize the function $\phi(x, y) = \sqrt{x^2 + y^2}$ with the constraint that $f(x, y) = (x-2)^2 + 2 - y = 0$.

Note that without the constraint, we know what to do. Find the critical points of the function that satisfy $Df(p) = 0$, and then use the second derivative test to determine whether the point p is a minimum or maximum.

Theorem. *Let $\phi, f : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions, and let $c \in \mathbb{R}$ be our constraint. Then if $p \in \mathbb{R}^n$ is a maximum or a minimum of the restriction $\phi|_{f=c}$, then*

$$\nabla\phi(p) = \lambda\nabla f(p)$$

λ is called the ***Lagrange multiplier.***

Instead of looking for maxima and minima of ϕ, the theorem says that we can look for solutions to the above equation. Practically, the theorem will result in finitely many points, at which point you can manually check if they are minima or maxima.

Examples

- Consider the above example. In practice, its usually easier to work with the distance function squared. Local extrema of the distance function squared are also local extrema of the distance function. We have

  $$D\phi(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix} Df(x, y) \qquad = \begin{pmatrix} 2(x-2) & -1 \end{pmatrix}$$

  So solving the system

  $$\begin{cases} 2\lambda x = 2(x-2) \\ 2\lambda y = -1 \\ y = (x-2)^2 + 2 \end{cases}$$

  yields our candidates for minima and maxima.

[6] We discussed the linear version of this problem last semester. Given a subspace $W \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$, which point of W is closest to p? We saw that the *orthogonal projection* of p onto W is the closest such point. Lagrange multipliers are a nonlinear version of this problem.

4/3/2019 - Implicit and inverse function theorems

Implicit function theorem

We will work in a slightly more general situation than last time today. Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ be a C^1 function, and let $p \in \mathbb{R}^n \times \mathbb{R}^k$ with $c = f(p)$. We have the level set

$$\chi_c = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : f(x, y) = c\}$$

We would like to express χ_c near p as the graph of a function $g : \mathbb{R}^n \to \mathbb{R}^k$. The implicit function theorem will provide the conditions necessary to guarantee that this is possible.

Let $\frac{\partial f}{\partial x}(p) \in M_{k\times n}(\mathbb{R})$ be the matrix with entries $\frac{\partial f}{\partial x}(p)_{ij} = \frac{\partial f_i}{\partial x_j}(p)$. Similarly let $\frac{\partial f}{\partial y}(p) \in M_{k\times k}(\mathbb{R})$ be the matrix with entries $\frac{\partial f}{\partial y}(p)_{ij} = \frac{\partial f_i}{\partial y_j}(p)$. Then altogether we have

$$Df(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}$$

We can now state a more general version of the implicit function theorem.

Theorem (Implicit Function theorem)**.** *Let f and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ be defined as above, and let $p \in \mathbb{R}^n \times \mathbb{R}^k$. Then if the square matrix $\frac{\partial f}{\partial y}(p)$ is invertible, there exists an open neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^k$ of p and a unique C^1 function $g : U \to V$ such that*

$$\chi_c \cap (U \times V) = \mathit{graph}(g)$$

Last time we stated the implicit theorem in the case where $k = 1$. Then the condition that $\frac{\partial f}{\partial y}(p)$ is invertible just means that the single partial derivative $\frac{\partial f}{\partial y}(p)$ is nonzero.

Examples

- Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined by

  $$f(x, y_1, y_2) = (x^2(y_1 + y_2), y_1 \cos(x - 1) - y_2)$$

  Then for the point $p = (1, 1, 1)$ and $c = f(p) = (2, 0)$, we have

  $$Df(x, y_1, y_2) = \left(\begin{array}{ccc} \underbrace{\begin{array}{c} 2x(y_1 + y_2) \\ -y_1 \sin(x - 1) \end{array}}_{\frac{\partial f}{\partial x}} & \begin{array}{c} x^2 \\ \cos(x - 1) \end{array} & \begin{array}{c} x^2 \\ -1 \end{array} \end{array} \right)$$

  Then

  $$\frac{\partial f}{\partial y}(p) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

  This matrix is invertible, so this implicit function theorem implies that we can solve the system

  $$\begin{cases} x^2(y_1 + y_2) = 2 \\ y_1 \cos(x - 1) - y_2 = 0 \end{cases}$$

to express y_1, y_2 as functions of x. In this case, we can explicit solve for

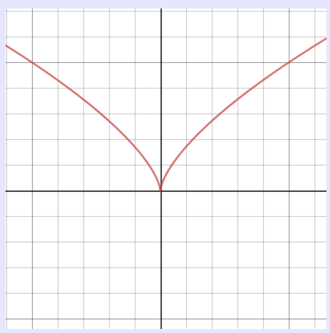
$$\begin{cases} y_1 = \frac{2}{x^2(\cos(x-1)+1} \\ y_2 = \frac{2\cos(x-1)}{x^2(\cos(x-1)+1)} \end{cases}$$

We have the function

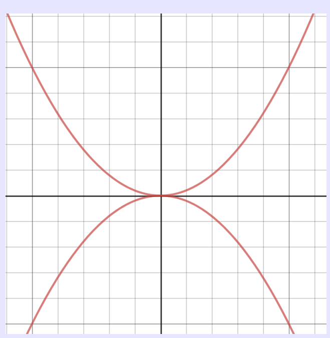
$$g(x) = \left(\frac{2}{x^2(\cos(x-1)+1}, \frac{2\cos(x-1)}{x^2(\cos(x-1)+1)} \right)$$

Near the point p, the level set χ_c is given as the graph of g.

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2 - y^3$. Then for the point $p = (0, 0)$ and $c = f(p) = 0$, we have
$$Df(p) = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
The implicit function theorem does not apply, but the equation $x^2 - y^3 = 0$ means $y = x^{2/3}$. So the level set of f is expressible as the graph of the function $g(x) = x^{2/3}$, but g is not differentiable.



- Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $y^2 - x^4$. Then for the point $p = (0, 0)$ and $c = f(p) = 0$, we have
$$Df(p) = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
The implicit function theorem does not apply, but the equation $y^2 - x^4 = 0$ means $y = \pm x^2$. For the function $g(x) = x^2$, we indeed have $f(x, g(x)) = 0$ for all x, but $\chi_0 \neq \text{graph}(g)$ around $(0, 0)$ (we are missing half of the solutions).



Inverse function theorem

We can use the implicit function theorem to prove the inverse function theorem.

Theorem (Inverse function theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function, and let $p \in \mathbb{R}^n$. If $Df(p)$ is invertible, then there exists an open neighborhood $U \subset \mathbb{R}^n$ of p such that $f|_U : U \to f(U)$ is injective and the inverse $g : f(U) \to U$ is C^1.*

Before proving this, we will introduce some additional terminology.

Definition. *A function $f : U \to V$ between open subsets of $\mathbb{R}^n$ is a* ***diffeomorphism*** *if f is bijective, C^1, and its inverse is C^1.*

A diffeomorphism is the analogue of an isomorphism, but for open subsets of $\mathbb{R}^n$. We'll now prove the inverse function theorem assuming the implicit function theorem, which we will prove next time.

Proof. Let $q = f(p)$, and consider the function

$$\begin{aligned} F : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \\ (x, y) &\mapsto f(x) - y \end{aligned}$$

F is 0 precisely when $f(x) = y$. In particular, note that $F(p, q) = 0$. We have

$$DF(x, y) = \begin{pmatrix} Df(x) & -I \end{pmatrix}$$

$$DF(p, q) = \begin{pmatrix} Df(p) & -I \end{pmatrix}$$

where $-I$ is the $n \times n$ identity matrix times -1. By assumption, $Df(p)$ is invertible, so the implicit function theorem implies that there is an open neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^n$ and a map $g : V \to U$ so that F is the graph of g. In other words, we have

$$0 = F(g(y), y) = f(g(y)) - y$$

for all y. This shows $f \circ g$ is the identity on V. However, g still may not be a bijection, as it may not map to all of U. But we can fix this by taking $U_1 = f^{-1}(V) \cap U$.

Then $g : V \to U_1$ is inverse to f. The image of g is indeed contained in U_1, as for $y \in V$ we have $f(g(y)) = y$, which implies $g(y) \in f^{-1}(V) \cap U = U_1$. We know that g is a right-inverse to f, but we must also show $g \circ f$ is the identity on U_1. Let $x \in U_1$. By definition of g, we know $F(g(f(x)), f(x)) = 0$. However, $F(x, f(x)) = 0$ as well. g is the unique by the implicit function theorem, which means that $g(f(x))$ is the unique point such that $F(g(f(x)), f(x)) = 0$. This yields $g(f(x)) = x$. ☐

The main idea of the proof is to use the implicit function theorem to come up with a function g and then verify that g is indeed a local inverse to f.

Application of the inverse function theorem

Recall that if $T : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a surjective linear map, then there exists an invertible linear $S : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ such that $T \circ S : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection given by sending (x, y) to y. The rank theorem is a nonlinear version of this result.

Theorem (Rank theorem)**.** *If $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a C^1 function, and at the point $p \in \mathbb{R}^{n+m}$ $Df(p)$ is surjective, then there exists a diffeomorphism*

$$G : \mathbb{R}^{n+m} \supset V \to U \ni p$$

such that $f \circ G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection given by sending (x, y) to y.

In other words, if the derivative of a map f at a point is surjective, then up to a change of coordinates the map looks like a projection onto $\mathbb{R}^n$.

Proof. Viewing $\mathbb{R}^m \times \mathbb{R}^n$, we have

$$Df(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}$$

Since surjectivity means that the columns of the matrix span $\mathbb{R}^n$, by reordering the columns we can assume $\frac{\partial f}{\partial y}(p)$ is surjective and hence invertible. Now consider

$$\begin{aligned} F : \mathbb{R}^m \times \mathbb{R}^n &\to \mathbb{R}^m \times \mathbb{R}^n \\ (x, y) &\mapsto (x, f(x, y)) \end{aligned}$$

Then the derivative of F is

$$\begin{aligned} DF(p) &= \begin{pmatrix} \frac{\partial F_1}{\partial x}(p) & \frac{\partial F_1}{\partial y}(p) \\ \frac{\partial F_2}{\partial x}(p) & \frac{\partial F_2}{\partial y}(p) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix} \end{aligned}$$

Then $DF(p)$ is invertible, since $\frac{\partial f}{\partial y}(p)$ is an invertible $(n \times n)$-submatrix.

The inverse function theorem implies that there is a neighborhood $U \subset \mathbb{R}^m \times \mathbb{R}^n$ of p such that $F : U \to F(U)$ has inverse $G : F(U) \to U$. Since $F(x, y) = (x, f(x, y))$ is the identity in the first coordinate, this implies that G is also the identity in the first coordinate, namely that $G(x, y) = (x, g(x, y))$ for some function $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. We know that $F \circ G$ is the identity, so this yields

$$\begin{aligned} (x, y) &= F \circ G(x, y) \\ &= F(x, g(x, y)) \\ &= (x, f(x, g(x, y))) \end{aligned}$$

Hence $y = f(x, g(x, y))$, so

$$f \circ G(x, y) = f(x, g(y)) = y$$

So G is the desired diffeomorphism that makes f look like the projection onto the second coordinate. ∎

4/8/2019 - Manifolds and Lagrange multipliers

Implicit and inverse function theorems

Recall the inverse function theorem, which says that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function such that $DF(p)$ is invertible, then F is invertible near p with a C^1 inverse.

Also recall the implicit function theorem, which says that given a C^1 function $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ and a point $p \in \mathbb{R}^n \times \mathbb{R}^k$ with $c = f(p)$, if when we write

$$Df(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}$$

the submatrix $\frac{\partial f}{\partial y}(p)$ is invertible, then there exists an open neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^k$ and a C^1 map $g : U \to V$ such that $f(x, g(x)) = c$ for all $x \in U$.

We had the rank theorem as well, which states that given a C^1 function $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ with $Df(p)$ surjective, there exists a diffeomorphism $H : V \to U$, where $U, V \subset \mathbb{R}^k$ are open sets with $p \in U$, such that

$$\begin{aligned} V &\xrightarrow{H} U \xrightarrow{f} \mathbb{R}^k \\ (x, y) &\longmapsto y \end{aligned}$$

We proved the second two of the above theorems using the implicit function theorem. We'll now return to prove the implicit function theorem.

Proof. Consider the function

$$\begin{aligned} F : \mathbb{R}^n \times \mathbb{R}^k &\to \mathbb{R}^n \times \mathbb{R}^k \\ (x, y) &\mapsto (x, f(x)) \end{aligned}$$

F has derivative

$$DF(p) = \begin{pmatrix} I & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}$$

By assumption, $\frac{\partial y}{\partial y}(p)$ is invertible, so $DF(p)$ is also invertible.

Then the inverse function theorem implies there exists an open neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^k$ of p such that $F|_U : U \to F(U)$ is invertible with a C^1 inverse $H : F(U) \to U$. Since $F(x, y) = (x, f(x, y))$, necessarily $H(x, y) = (x, h(x, y))$ for some function $h : F(U) \to U \cap (\{0\} \times \mathbb{R}^k)$ (as if F leaves the first coordinate unchanged, its inverse H also leaves the first coordinate unchanged).

Write $p = (a, b) \in \mathbb{R}^n \times \mathbb{R}^k$. Then $F(p) = (a, c)$. Since $F \circ H$ is the identity, we have

$$\begin{aligned} (x, c) &= F \circ H(x, c) \\ &= F(x, h(x, c)) \\ &= \big(x, f(x, h(x, c))\big) \end{aligned}$$

Hence this implies $f(x, h(x, c)) = c$ for all x near a, which completes the proof. $\square$

Note that we used the implicit function theorem to prove the inverse function theorem. So we have shown that these two theorems imply each other. In fact, the same argument proves the rank theorem as well, where H from the above proof is the diffeomorphism we are looking for. This is because for all d near c, we have

$$f \circ H(x, d) = f(x, h(x, d)) = d$$

which means H is the diffeomorphism that makes f into a projection.

Note that the rank theorem gives a complete picture of f near p. The idea is that if $Df(p)$ is surjective, then the level sets of f near p look like a stack of planes orthogonal to the gradient vector.

Lagrange multipliers

These theorems have a very useful application.

Theorem (Lagrange multipliers theorem)**.** *Let $\phi, f : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions. Fix $c \in \mathbb{R}$, and let $X = f^{-1}(c)$. Assume $Df(x) \neq 0$ for all $x \in X$. If p is a local maximum or minimum of the restriction $\phi|_X$, then*

$$\nabla\phi(p) = \lambda\nabla f(p)$$

for some $\lambda \in \mathbb{R}$.

Examples

- We can use Lagrange multipliers to find the maximum or minimum distance to the origin on the curve $X = f^{-1}(9)$, where

  $$f(x, y) = x^{14} + 35xy^2 + 2x^6y + 20x^5y^5 + 10x^2y + y^{12}$$

  In this case, ϕ is the function given by taking the distance to the origin. We are looking for a local minimum or maximum of ϕ restricted to the level set X.

The above example suggests why the converse of the Lagrange multipliers theorem is false. Just consider the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\phi(x, y) = y$ restricted to the 0-level set of the function $f(x, y) = x^3$.

We will first need an important lemma.

Lemma. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1. Fix $c \in \mathbb{R}$, and define $X = f^{-1}(c)$. Take a C^1 function $\gamma : (-1, 1) \to X$. Then if $p = \gamma(0)$, the vectors $\nabla f(p)$ and $\gamma'(0)$ are orthogonal.*

In other words, the lemma is saying that the level set of a function is orthogonal to the gradient. The function γ is just a curve on the level set.

Examples

- Let $f(x,y,z) = x^2 + y^2 + z^2$. Then the level set $X = f^{-1}(1)$ is the unit sphere. The derivative of any curve γ that lies on the surface of the sphere will always be orthogonal to the gradient of f, which points radially outwards.

Proof. The proof follows immediately from the chain rule. Since the image of γ is always in the level set, we know $c = f \circ \gamma(t)$ for all $t \in (-1, 1)$. Then differentiation yields

$$\begin{aligned} 0 &= D(f \circ \gamma)(t) \\ &= Df(\gamma(t)) \cdot \gamma'(t) \\ 0 &= Df(p) \cdot \gamma'(0) \end{aligned}$$

taking $t = p$. Where $\cdot$ is the dot product on $\mathbb{R}^n$, as the matrices are just row/column vectors (so matrix multiplication/composition is just given by taking the dot product). □

We can now prove the Lagrange multipliers theorem.

Proof. Suppose $p \in X$ is a local minimum or maximum. If $D\phi(p) = 0$, then take $\lambda = 0$, and we are done.

Assume $D\phi(p) \neq 0$. We also know $Df(p) \neq 0$ by assumption. Suppose for contradiction that $\nabla\phi(p)$ and $\nabla f(p)$ are not proportional. The rank theorem says that the level sets of ϕ look like a stack of planes orthogonal to $\nabla f(p)$. The idea is that if these two vectors are not parallel, then by moving along X around p we can find a larger or smaller value. So p is not a local extrema, which is a contradiction.[7] □

On the next homework you will prove the spectral theorem[8] using Lagrange multipliers.

Manifolds

Definition. *Informally, a **k-dimensional manifold** in $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ that is locally the graph of a function from R^k to R^{n-k}.*

Examples

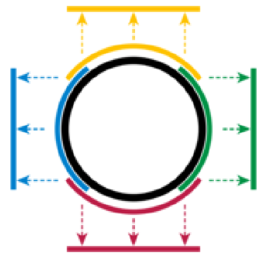
- The circle $M = \{(x,y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ is a 1-dimensional manifold. Any point on M lies on the graph of one of the functions

$$y = \pm\sqrt{1 - x^2}$$

$$x = \pm\sqrt{1 - y^2}$$

[7] This proof relies heavily on relevant diagrams and illustrations. Email me if you are confused and including some illustrations would help.

[8] If $A \in M_n(\mathbb{R})$ is a symmetric matrix, then there exists an orthonormal basis $u_1, \ldots, u_n \in \mathbb{R}^n$ of eigenvectors such that $Au_i = \lambda_i u_i$.



We can give a more formal definition in the case of 1-dimensional manifolds.

Definition. *A **1-dimensional manifold** in $\mathbb{R}^2$ is a subset $M \subset \mathbb{R}^2$ such that for all $p \in M$, there exists an open rectangle $Q = (a,b) \times (c,d)$ around p and a C^1 function $h : (a,b) \to \mathbb{R}$ or $h : (c,d) \to \mathbb{R}$ such that*

$$M \cap Q = \mathit{graph}(h)$$

In general, we can ask when the level set of a function $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a manifold. We can answer this question using some of the theorems we have been discussing.

Theorem (Manifold Recognition theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a C^1 function, and fix $c \in \mathbb{R}^{n-k}$. If $Df(p)$ is surjective for all $p \in f^{-1}(c)$, then the level set $f^{-1}(c)$ is a k-dimensional manifold in $\mathbb{R}^n$.*

The proof of this theorem is just the implicit function theorem.

Examples

- The configuration space of a collection of linked rods is a manifold. We can describe the collection of possible positions of each rod. Let r_1 and r_2 be the lengths of the two rods. We are fixing one end of the first rod to the origin, and fixing one end of the second rod to the other end of the first rod.

  Let (x_1, y_1) be the end of the first rod/beginning of the second rod and let (x_2, y_2) be the end of the second rod. (x_1, y_1) always lies on the circle of radius r_1 centered at the origin, so we have
  $$f_1(p) = x_1^2 + y_1^2 = r_1^2$$
  (x_2, y_2) always lies a distance of r_2 from (x_1, y_1), so we have
  $$f_2(p) = (x_2 - x_1)^2 + (y_2 - y_1)^2 = r_2^2$$
  If we define the function
  $$\begin{aligned} f : \mathbb{R}^4 &\to \mathbb{R}^2 \\ p &\mapsto (f_1(p), f_2(p)) \end{aligned}$$
  Then the preimage $f^{-1}(r_1^2, r_2^2)$ consists of all the possible configurations of the linked rods. It's not too hard to see that $Df(p)$ is surjective at each $p \in M$, which means that the configuration space is a 2-dimensional manifold in $\mathbb{R}^4$.

  Given some thought, one can see $M \simeq S^1 \times S^1$ (which is the torus!).

4/10/2019 - Stokes theorem, Linear and differential forms

Forms on vector spaces

Today we will start a new topic. The goal is to cover Stokes' theorem, which is a vast generalization of the fundamental theorem of calculus with many applications. The theorem is encapsulated in the equation

$$\int_c d\omega = \int_{\partial c} \omega$$

It will take us some time to understand all the parts of this equation. ω is a *differential k-form*, $d\omega$ is the *exterior derivative* of ω, c is a *k-cube* in $\mathbb{R}^n$, and ∂c is the *boundary* of c.

In the case when $k = 1$, this equation becomes

$$\int_{[0,1]} f' = f(1) - f(0)$$

We will speak about differential forms today. Recall that the determinant

$$\det : \mathbb{R}^k \times \ldots \times \mathbb{R}^k \to \mathbb{R}$$

is the unique, multilinear,[9] alternating[10] function on $(\mathbb{R}^k)^k$ such that

$$\det(e_1, \ldots, e_k) = 1$$

We can view the determinant as a function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ of the rows of a matrix. In this context, the determinant should be understood as a measure of the signed area of the parallelogram spanned by the rows of a matrix.

k-forms on $\mathbb{R}^n$ will give us a notion of k-dimensional volume in $\mathbb{R}^n$, for $k \leq n$.

Definition. *A **k-form** on $\mathbb{R}^n$ is a multilinear, alternating function*

$$\phi : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k\ times} \to \mathbb{R}$$

Denote the set of all k-forms on $\mathbb{R}^n$ by

$$\Lambda^k(\mathbb{R}^n) = \{k\text{-forms on } \mathbb{R}^n\}$$

Then $\Lambda^k(\mathbb{R}^n)$ is a real vector space in the obvious way.[11]

---

[9] Multilinear means that if we fix $k-1$ of the entries, the resulting function from $\mathbb{R}^k$ to $\mathbb{R}$ is linear.

[10] Alternating means that the determinant changes signs if you swap two entries, namely that

$$\det(e_1, \ldots, e_i, \ldots, e_j, \ldots, e_k) = -\det(e_1, \ldots, e_j, \ldots, e_i, \ldots, e_k)$$

[11] Meaning with pointwise addition and scalar multiplication, given by

$$(\phi + \psi)(v_1, \ldots, v_k) = \phi(v_1, \ldots, v_k) + \psi(v_1, \ldots, v_k)$$
$$(a\phi)(v_1, \ldots, v_k) = a\phi(v_1, \ldots, v_k)$$

Examples

- A 1-form on $\mathbb{R}^n$ is a linear map $\phi : \mathbb{R}^n \to \mathbb{R}$. Hence

$$\Lambda^1(\mathbb{R}^n) = L(\mathbb{R}^n, \mathbb{R})$$

Elementary k-forms

Fix indices $1 \le i_1, \ldots, i_k \le n$. Define a k-form on $\mathbb{R}^n$ with the following procedure.

Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, form the matrix

$$\begin{pmatrix} v_{11} & \ldots & v_{k1} \\ \ldots & & \ldots \\ v_{1n} & \ldots & v_{kn} \end{pmatrix}$$

This is an $n \times k$ matrix. Use the above indices to choose the following submatrix

$$M(v_1, \ldots, v_k) = \begin{pmatrix} v_{1i_1} & \ldots & v_{k,i_1} \\ \ldots & & \ldots \\ v_{1,i_k} & \ldots & v_{k,i_k} \end{pmatrix}$$

Then the map

$$\phi(v_1, \ldots, v_k) = \det(M(v_1, \ldots, v_k))$$

is alternating and multilinear because the determinant is. Denote this k-form by

$$dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

Examples

- On the vector space $\mathbb{R}^4$, the 1-form dx_i acts by

$$dx_i \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v_i$$

- On the vector space $\mathbb{R}^4$, the 2-form $dx_1 \wedge dx_3$ acts by

$$dx_1 \wedge dx_3 \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \\ 5 & 9 \end{pmatrix} = \det \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} = -8$$

- On the vector space $\mathbb{R}^4$, the 3-form $dx_3 \wedge dx_1 \wedge dx_4$ acts by

$$dx_3 \wedge dx_1 \wedge dx_4 \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \\ 5 & 9 & 13 \end{pmatrix} = \det \begin{pmatrix} 3 & 7 & 11 \\ 1 & 5 & 9 \\ 4 & 8 & 12 \end{pmatrix}$$

- On the vector space $\mathbb{R}^4$, the 4-form $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ is simply the determinant.

k-forms give a measure of k-dimensional volumes in $\mathbb{R}^n$. For example, the form $dx \wedge dy \in \Lambda^2(\mathbb{R}^3)$ acts by

$$dx \wedge dy \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix} = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

The two vectors (v_1, v_2, v_3) and (w_1, w_2, w_3) span a parallelogram in $\mathbb{R}^3$. The form $dx \wedge dy$ is the signed area of the projection of this parallelogram onto the xy-plane in $\mathbb{R}^3$.

For 3-dimensional volume on $\mathbb{R}^3$, the determinant is the only multilinear, alternating map (up to normalization). But for measuring 2-dimensional volume, there are many different elements of $\Lambda^2(\mathbb{R}^3)$. For example $dy \wedge dz$ is also a 2-form. $\Lambda^2(\mathbb{R}^3)$ is a vector space, so we have the sum $dx \wedge dy + dy \wedge dz$ as well.

The algebra of k-forms

Since det is alternating, we have relations introduced between forms in $\Lambda^k(\mathbb{R}^n)$. For example,

$$dx_1 \wedge dx_3 = -dx_3 \wedge dx_1$$

$$dx_1 \wedge dx_2 \wedge dx_1 = 0$$

$$dx_2 \wedge dx_3 \wedge dx_1 = dx_1 \wedge dx_2 \wedge dx_3$$

The second equation holds because the determinant vanishes if two rows are identical.

Definition. *If $1 \leq i_1 < \ldots < i_k \leq n$ is a sequence of strictly increasing indices, then*

$$dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

*is an **elementary k-form**.*

Theorem. *The elementary k-forms are a basis of $\Lambda^k(\mathbb{R}^n)$.*

Examples

- $\Lambda^1(\mathbb{R}^3)$ is spanned by the elementary forms dx, dy, and dz.
- $\Lambda^2(\mathbb{R}^3)$ is spanned by the elementary forms $dx \wedge dy$, $dy \wedge dz$, and $dx \wedge dz$.

Corollary. *The dimension of the vector space $\Lambda^k(\mathbb{R}^n)$ is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. The size of the basis is the number of ways to pick a strictly increasing sequence of indices between 1 and n, which is the number of ways to pick k distinct numbers between 1 and n, where the order doesn't matter. $\square$

Definition. *The **wedge product** is an operation*

$$\wedge : \Lambda^k(\mathbb{R}^n) \times \Lambda^\ell(\mathbb{R}^n) \to \lambda^{k+\ell}(\mathbb{R}^n)$$

It is defined by extending the obvious multiplication on elementary forms, forcing distributivity and associativity.

Examples

- We have

$$(8\,dx_1 \wedge dx_2 - 2\,dx_1 \wedge dx_3)(4\,dx_1 + 3\,dx_2) = 24\,dx_1 \wedge dx_2 \wedge dx_3$$

It follows from the theorem that $\Lambda^k(\mathbb{R}^n) = \{0\}$ if $k > n$, since it is impossible to pick a strictly increasing sequence of $n+1$ indices between 1 and n.

Note that the wedge product is not commutative (we saw this already with elementary forms). The wedge product is called *graded-commutative*, since

$$\phi \wedge \psi = (-1)^{k\ell}\psi \wedge \phi$$

where ϕ and ψ are k- and ℓ-forms, respectively.[12]

Some properties

- For every $\phi \in \Lambda^1(\mathbb{R}^3)$, $\phi \wedge \phi = 0$. To see this, write

  $$\phi = a\,dx + b\,dy + c\,dz$$

  Then

  $$(a\,dx + b\,dy + c\,dz)(a\,dx + b\,dy + c\,dz) = 0$$

  since the terms cancel in pairs.

- For every $\phi \in \Lambda^2(\mathbb{R}^3)$, $\phi \wedge \phi = 0$. This follows from a similar example as above, but one can also argue more directly. $\phi \wedge \phi$ is a 4-form on $\mathbb{R}^3$, but since $\Lambda^4(\mathbb{R}^3) = \{0\}$ by the above remark, we have $\phi \wedge \phi = 0$.

[12] The exponent of $k\ell$ comes from the fact that to rearrange $\phi \wedge \psi$ to $\psi \wedge \phi$, we have to move k terms past ℓ terms, which requires $k\ell$ swaps.

However, it is not always true that $\phi \wedge \phi = 0$. For the form $\phi = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ in $\Lambda^2(\mathbb{R}^4)$, we have $\phi \wedge \phi = 2\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

Differential forms

Definition. *Let $U \subset \mathbb{R}^n$ be open. A **differential k-form** on U is a smooth function*

$$\omega : U \to \Lambda^k(\mathbb{R}^n)$$

Denote the set of all differential k-forms on U by

$$\Omega^k(U) = \{\textit{differential k-forms on } U\}$$

Examples

- By convention, $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$. So

$$\Omega^0(\mathbb{R}^n) = \{\text{smooth functions } U \to \mathbb{R}\}$$

- Since $\Lambda^k(\mathbb{R}^n)$ is spanned by the elementary forms, we can write any $\omega \in \Omega^2(\mathbb{R}^3)$ as

$$\omega(p) = f(p)\, dx \wedge dy + g(p)\, dx \wedge dz + h(p)\, dy \wedge dz$$

  where $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ are smooth and unique.

Note that a k-form takes k vectors in $\mathbb{R}^n$ and yields a real number. A differential k-form takes a point $p \in \mathbb{R}^n$, k vectors in $\mathbb{R}^n$, and yields a real number

$$\omega(p)(v_1, \ldots, v_k)$$

as $\omega(p)$ is a k-form.

Examples

- Take $\omega = dx$ and $\eta = x\, dy$ in $\Omega^1(\mathbb{R}^2)$. For the points $p = (0,0)$ and $q = (2,2)$ and vectors $u = (0,1)$ and $v = (2,3)$, we have

$$\eta(q)(u) = 2\, dy \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2$$

$$\eta(p)(u) = 0\, dy \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\omega(q)(v) = dx \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2$$

$$\omega(p)(v) = dx \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2$$

Viewing the derivative as a 1-form

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, for $p \in \mathbb{R}^n$ the linear map $Df(p) : \mathbb{R}^n \to \mathbb{R}$ is an element of $\Lambda^1(\mathbb{R}^n)$. Then we can define the 1-form

$$\begin{aligned} df : \mathbb{R}^n &\to \Lambda^1(\mathbb{R}^n) \\ p &\mapsto Df(p) \end{aligned}$$

If we look at df with explicit coordinates, then

$$df(p) = \sum_{i=1}^n D_i f(p)\, dx_i$$

1-forms and vector fields

Any $\omega \in \Omega^1(\mathbb{R}^n)$ can be written

$$\omega(p) = \sum_{i=1}^n F_i(p)\, dx_i$$

where $F_i : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. If we define

$$\begin{aligned} F : \mathbb{R}^n &\to \mathbb{R}^n \\ p &\mapsto (F_1(p), \ldots, F_n(p)) \end{aligned}$$

then

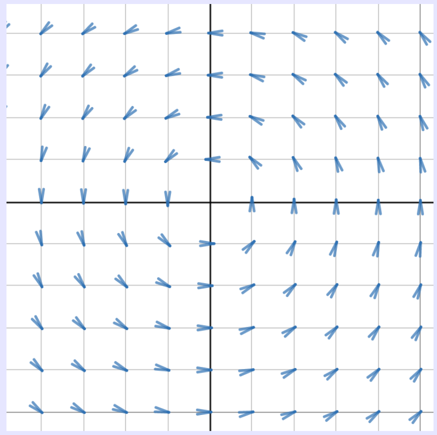
$$\begin{aligned} \omega(p)(v) &= \left(\sum_{i=1}^n F_i\, dx_i \right) \begin{pmatrix} v_1 \\ \cdots \\ v_n \end{pmatrix} \\ &= \sum F_i(p)\, dx_i \\ &= F(p) \cdot v \end{aligned}$$

where $\cdot$ is the dot product.

Examples

- Consider the 1-form
$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$
in $\Omega^1(\mathbb{R}^2 \setminus \{0\})$ (as this form is not defined at the origin). Here
$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

This is the vector field



Given $c : [0,1] \to \mathbb{R}^2 \setminus \{0\}$, we can compute

$$\int_0^1 F(c(t)) \cdot c'(t)\, dt$$

For example, if $c(t) = (1+t, 0)$ we have $c'(t) = (1,0)$ and $F(c(t)) = (0, 1/(1+t))$. Then $F(c(t)) \cdot c'(t) = 0$ always so the integral over c is 0.

In contrast, if $c(t) = (\cos(2\pi t), \sin(2\pi t))$, then the integral of ω over c is 2π.

Hence we can see that this form has a nice geometric interpretation. It is the *winding number*, and it measures how many times a curve in $\mathbb{R}^2 \setminus \{0\}$ wraps around the origin.

4/15/2019 - Integration of differential forms

Linear forms on $\mathbb{R}^n$

We are working towards Stokes' theorem, which says that if ω is a differential k-form and $c : [0,1]^{k+1} \to \mathbb{R}^n$ is a cube, then

$$\int_{\partial c} \omega = \int_c d\omega$$

Last time we defined differential forms, and today we will define what it means to integrate over a differential form.

Recall that a linear k-form on $\mathbb{R}^n$ is a multilinear, alternating function

$$\phi : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k \text{ times}} \to \mathbb{R}$$

We will denote $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ by $(\mathbb{R}^n)^{\times k}$. We defined the elementary k-form

$$dx_J = dx_{j_1} \wedge \ldots \wedge dx_{j_k}$$

given a collection of strictly increasing indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_1 < j_2 < \ldots < j_k \leq n$. This a function on $(\mathbb{R}^n)^{\times k}$. It takes a k vectors $(v_1, \ldots, v_k)$, each expressed in terms of a basis

$$v_i = v_{i1} e_1 + \ldots + v_{in}$$

and yields

$$dx_J(v_1, \ldots, v_k) = \det \begin{pmatrix} v_{1,j_1} & \ldots & v_{k,j_1} \\ \ldots & & \ldots \\ v_{1,j_k} & \ldots & v_{k,j_k} \end{pmatrix}$$

We packaged all of the linear k-forms on $\mathbb{R}^n$ into the vector space $\Lambda^k(\mathbb{R}^n)$. We are particularly interested in the elementary k-forms because they form a basis for the vector space $\Lambda^k(\mathbb{R}^n)$.

Examples

- Given any 1-form $\phi \in \Lambda^1(\mathbb{R}^n) = L(\mathbb{R}^n, \mathbb{R})$, we have

$$\begin{aligned} \phi(v) &= \phi\left(\sum_{i=1}^n v_i e_i\right) \\ &= \sum_{i=1}^n v_i \phi(e_i) \\ &= \sum_{i=1}^n \phi(e_i)\, dx_i(v) \end{aligned}$$

  where dx_i is the 1-form on $\mathbb{R}^n$ that simply picks out the ith row of the vector v. This demonstrates that every 1-form is in the span of the elementary 1-forms. Furthermore, writing

$$w = \begin{pmatrix} \phi(e_1) \\ \dots \\ \phi(e_n) \end{pmatrix}$$

shows that the form ϕ is given by the standard inner product

$$\phi(v) = v \cdot w$$

Theorem. *The elementary k-forms are a basis for $\Lambda^k(\mathbb{R}^n)$.*

Proof. For concreteness, we'll consider the case when $k = 2$ and $n = 3$. The general case differs only in the notation. We want to show that $\Lambda^2(\mathbb{R}^3)$ has basis

$$\begin{aligned} dx_{12} &= dx_1 \wedge dx_2 \\ dx_{13} &= dx_1 \wedge dx_3 \\ dx_{23} &= dx_2 \wedge dx_3 \end{aligned}$$

We'll show that these vectors span and are linearly independent. Let $\phi \in \Lambda^2(\mathbb{R}^3)$ be an arbitrary 2-form on $\mathbb{R}^3$. We have

$$\begin{aligned} \phi(v, w) &= \phi\left(\sum_{i=1}^{3} v_i e_e, \sum_{j=1}^{3} w_j e_j \right) \\ &= \sum_{1 \le i,j \le 3} v_i w_j \phi(e_i, e_j) \end{aligned}$$

using the multilinearity of ϕ. Some of these terms will be zero and $\phi(e_i, e_j) = -\phi(e_j, e_i)$, so we can adjust indices for

$$\begin{aligned} \phi(v, w) &= \sum_{1 \le i < j \le 3} (v_i w_j - v_j w_i) \phi(e_i, e_j) \\ &= \sum_{1 \le i < j \le 3} \phi(e_i, e_j)\, dx_{ij}(v, w) \\ &= \phi(e_1, e_2)\, dx_{12} + \phi(e_1, e_3)\, dx_{13} + \phi(e_2, e_3)\, dx_{23} \end{aligned}$$

The second line is justified by definition[13] of the elementary form dx_{ij}.

To show these elementary 2-forms are linearly independent, suppose

$$a\, dx_{12} + b\, dx_{13} + c\, dx_{23} = 0$$

Evaluating this form on (e_1, e_2) yields $a = 0$. Similarly, $b = c = 0$, so the elementary 2-forms are linearly independent. $\square$

[13]For example,

$$dx_{23}\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) = \det \begin{pmatrix} v_2 & w_2 \\ v_3 & w_3 \end{pmatrix} = v_2 w_3 - v_3 w_2$$

Integration of differential k-forms

Let $U \subset \mathbb{R}^n$ be an open set. Recall that a differential k-form is a smooth function

$$\omega : U \to \Lambda^k(\mathbb{R}^n)$$

We denote the collection of differential k-forms by $\Omega^k(U)$. For each $p \in U$, we have that $\omega(p) \in \Lambda^k(\mathbb{R}^n)$. Since the vector space $\Lambda^k(\mathbb{R}^n)$ is spanned by the elementary k-forms, we can write

$$\omega(p) = \sum_{J=(j_1,\ldots,j_k)} f_J(p)\, dx_J$$

where each $f_J : U \to \mathbb{R}$ is a smooth function. We can motivate integration by examining the case of 1-forms.

Integration of 1-forms

Let $\omega \in \Omega^1(\mathbb{R}^n)$. We can write

$$= f_1\, dx_1 + \ldots + f_n\, dx_n$$

Package the $f_i : \mathbb{R}^n \to \mathbb{R}$ functions together for a smooth vector field

$$\begin{aligned} F : \mathbb{R}^n &\to \mathbb{R}^n \\ p &\mapsto (f_1(p), \ldots, f_n(p)) \end{aligned}$$

Then we have

$$\begin{aligned} \omega(p)(v) &= \sum_{i=1}^{n} f_i(p)\, dx_i \begin{pmatrix} v_1 \\ \ldots \\ v_n \end{pmatrix} \\ &= \sum_{i=1} f_i(p) v_i \\ &= F(p) \cdot v \end{aligned}$$

Definition. *Given a smooth function $c : [0,1] \to \mathbb{R}^n$, the* ***line integral*** *of ω over c is defined*

$$\int_c \omega = \int_0^1 \omega(c(t))(c'(t))\, dt$$

Examples

- Consider the case when $n = 2$, and write dx, dy instead of dx_1, dx_2. Let $\omega = dx = 1\, dx + 0\, dy$, the constant differential form. Then

  $$F(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Given a smooth function $c : [0,1] \to \mathbb{R}^2$, we have

$$c(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$
$$c'(t) = \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix}$$

Then

$$\begin{aligned} \int_c dx &= \int_0^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} dt \\ &= \int_0^1 c_1'(t)\, dt \\ &= c_1(1) - c_1(0) \end{aligned}$$

So integrating dx along c yields the net x-variation of c.

- More generally, integrating an arbitrary 1-form over c looks like

$$\int_c f\, dx + g\, dy$$

This measures the net 'weighted x- and y-variation' along a curve.

- Last time, we saw that the form

$$\omega = \frac{1}{x^2+y^2}\Big(-y\, dx + x\, dy \Big)$$

measures the net winding of c around 0. At a general point, the vector field for ω points along the circles centered at the origin:

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

For example, the curve

$$\begin{aligned} c : [0,1] &\to \mathbb{R}^2 \setminus \{0\} \\ t &\mapsto (\cos(2\pi t), \sin(2\pi t)) \end{aligned}$$

Concretely, we can compute

$$\begin{aligned} \int_c \omega &= \int_0^1 \begin{pmatrix} -\sin(2\pi t) \\ \cos(2\pi t) \end{pmatrix} \cdot \begin{pmatrix} -2\pi \sin(2\pi t) \\ 2\pi \cos(2\pi t) \end{pmatrix} \\ &= \int_0^1 2\pi(sin^2(2\pi t) + \cos^2(2\pi t))\, dt \\ &= 2\pi \end{aligned}$$

The fundamental theorem of line integrals

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we have the differential 1-from

$$df = D_1 f\, dx_1 + \ldots + D_n f\, dx_n$$

The vector field of df is

$$\begin{pmatrix} D_1 f \\ \ldots \\ D_n f \end{pmatrix} = \nabla f$$

Note that

$$df(p)(v) = Df(p)(v)$$

Integration of df over some curve $c : [0, 1] \to \mathbb{R}^n$ is

$$\begin{aligned} \int_c df &= \int_0^1 \nabla f(c(t))c'(t)\, dt \\ &= \int_0^1 Df(c(t))c'(t)\, dt \\ &= \int_0^1 (f \circ c)'(t)\, dt \\ &= f(c(1)) - f(c(0)) \end{aligned}$$

by the chain rule and the fundamental theorem of calculus, since $f \circ c$ is a function from $\mathbb{R}$ to $\mathbb{R}$. This is in fact a special case of Stokes' theorem.

Theorem (Fundamental theorem of line integrals)**.** *If $\omega \in \Omega^1(\mathbb{R}^n)$ is given by $\omega = df$ for some smooth $f : \mathbb{R}^n \to \mathbb{R}$, then*

$$\int_c \omega = f(c(1)) - f(c(0))$$

for any smooth curve $c : [0, 1] \to \mathbb{R}^n$.

In other words, the integral of a differential 1-form that arises as the derivative of a smooth function depends only on the endpoints of the curve.

Examples

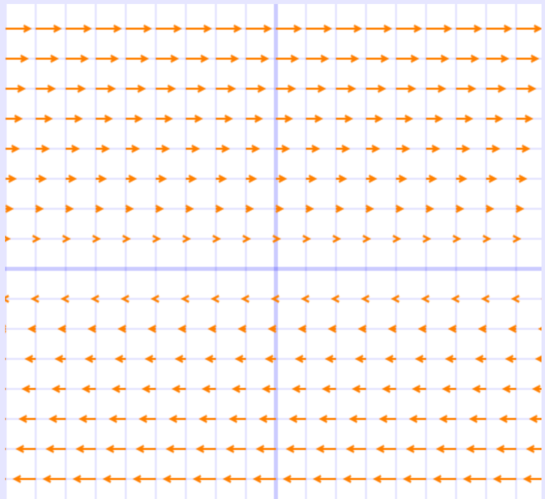
- Let $\omega \in \Omega^1(\mathbb{R}^2)$ be defined $\omega = y\, dx$. Consider the curves

$$\begin{aligned} c(t) &= (\cos(\pi t), \sin(\pi t)) \\ d(t) &= (\cos(\pi t), -\sin(\pi t)) \\ e(t) &= (1 - 2t, 0) \end{aligned}$$

  Recall that we have

$$\int_c \omega = \int_0^1 F(c(t)) \cdot c'(t)\, dt$$

where F is the vector field corresponding to ω. This vector field is given by $F(x, y) = (y, 0)$.



The integrals over the above curves are given by the dot product of the vector field with the derivative of the curve at each point. Then we have

$$\underbrace{\int_c \omega}_{<0} < \underbrace{\int_e \omega}_{=0} < \underbrace{\int_d \omega}_{>0}$$

Note that even though the the endpoints of these curves are equal, their integrals differ. This is because $\omega \neq df$ for all f.

We can define the integral over a differential k-form more generally.

Definition. *Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $c : I^k \to \mathbb{R}^n$ be a smooth map of the k-cube $I^k = [0,1]^k$. Write*

$$\omega = \sum_{J=(j_1,\ldots,j_k)} f_J \, dx_J$$

*The **integral** of ω over c is defined*

$$\int_c \omega = \sum_J \int_{I^k} (f_J \circ c) \, dx_J(Dc)$$

Examples

- Consider the case when $k = 2$ and $n = 3$, and take a differential form

  $$\omega = f_{12} \, dx_{12} + f_{13} \, dx_{13} + f_{23} \, dx_{23}$$

  Given a map

  $$\begin{aligned} c : I^2 &\to \mathbb{R}^3 \\ (s,t) &\mapsto (c_1(s,t), c_2(s,t), c_3(s,t)) \end{aligned}$$

We have

$$Dc = \begin{pmatrix} \frac{\partial c_1}{\partial s} & \frac{\partial c_1}{\partial t} \\ \frac{\partial c_2}{\partial s} & \frac{\partial c_2}{\partial t} \\ \frac{\partial c_3}{\partial s} & \frac{\partial c_3}{\partial t} \end{pmatrix}$$

Denote

$$\frac{\partial c_{ij}}{\partial(s,t)} = \begin{pmatrix} \frac{\partial c_i}{\partial s} & \frac{\partial c_i}{\partial t} \\ \frac{\partial c_j}{\partial s} & \frac{\partial c_j}{\partial t} \end{pmatrix}$$

Then

$$dx_{ij}(Dc) = \det\left(\frac{\partial c_{ij}}{\partial(s,t)}\right)$$

Then

$$\int_c \omega = \sum_{i<j} \int_{I^2} f_{ij}(c(s,t)) \det\left(\frac{\partial c_{ij}}{\partial(s,t)}\right)$$

If $\omega = dx_1 \wedge dx_2$, then $\int_c \omega$ measures the net area of the projection of c to the x_1x_2-plane.

4/17/2019 - Exterior derivatives and pullbacks

Exterior derivatives

The next ingredient we need for Stokes' theorem is the *exterior derivative*. Recall that a differential k-form $\omega \in \Omega^k(\mathbb{R}^n)$ is a smooth map

$$\omega : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$$

We can write ω as the linear combination of elementary k-forms for

$$\omega = \sum_J f_J \, dx_J$$

where $J = (j_1, \ldots, j_k)$ ranges over strictly increases sequences of indices $1 \le j_1 < \ldots < j_k \le n$. We can also understand a differential form ω as a map from k-cubes to $\mathbb{R}$ for

$$\begin{aligned} \omega : \{k\text{-cubes } c : I^k \to \mathbb{R}^n\} &\to \mathbb{R} \\ \left(c : I^k \to \mathbb{R}^n \right) &\mapsto \int_c \omega \end{aligned}$$

The integral is given by

$$\begin{aligned} \int_c \omega &= \int_{t \in I^k} \omega(c(t))(Dc(t)) \\ &= \sum_J \int_{I^k} f_J(c(t)) \, dx_J(D_1 c(t), \ldots, D_k c(t)) \end{aligned}$$

With this perspective, the differential form ω is a *functional*. Hence we have two ways of viewing a differential form $\omega \in \Omega^k(U)$ as a function of sorts. The first is that ω is a smooth function from U to $\Lambda^k(\mathbb{R}^n)$. The second is that ω is a map from the set of k-cubes to $\mathbb{R}$.

Examples

- The integral $\int_c \omega$ recovers our previous notion of the integral $\int_Q f$ of a function f over a closed rectangle Q. Given $f : Q \to \mathbb{R}$ where $Q \subset \mathbb{R}^n$ is a closed rectangle, let the n-cube $c : Q \to \mathbb{R}^n$ be the inclusion map. Also let $\omega = f \, dx_1 \wedge \ldots \wedge dx_n$. Then

$$\begin{aligned} \int_c \omega &= \int_{t \in Q} \omega(t)(I) \\ &= \int_{t \in Q} f(t) \, dx_1 \wedge \ldots \wedge dx_n(e_1, \ldots, e_n) \\ &= \int_Q f \end{aligned}$$

  since the derivative of the inclusion map c is always the identity matrix I.

- Let $\omega \in \Omega^2(\mathbb{R}^3)$ be given by

$$\omega = f_{12} \, dx_{12} + f_{13} \, dx_{13} + f_{23} \, dx_{23}$$

Then we have a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F = (f_{23}, -f_{13}, f_{12})$. Observe that

$$\omega(p)(u, v) = F(p) \cdot (u \times v)$$

where $\cdot$ is the dot product and $\times$ is the cross product. Then for a 2-cube $c : I^2 \to \mathbb{R}^3$, we have

$$\begin{aligned}\int_c \omega &= \int_{I^2} \omega(c(t))(D_1c(t), D_2c(t)) \\ &= \int_{I^2} F(c(t)) \cdot (D_1c(t) \times D_2c(t))\end{aligned}$$

This is sometimes called the *flux integral*, as the cross product $D_1c(t) \times D_2c(t)$ yields a vector normal to the surface that is the image of c. Then the integral is a measure of the extent to which the vector field is 'flowing through' the surface defined by c.

Today we will discuss differentiating differential k-forms.

Definition. *Let $U \subset \mathbb{R}^n$ be an open subset. The **exterior derivative** is a map*

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

If $\omega \in \Omega^k(U)$, write

$$\omega = \sum_J f_J \, dx_J$$

Then d is given by

$$d\omega = \sum_J \sum_{i=1}^n D_i f_J \, dx_i \wedge dx_J$$

When $k = 0$, note that we have the following familiar properties.

1. d acts on $f \in \Omega^0(U)$ by
$$df = D_1 f \, dx_1 + \ldots + D_n f \, dx_n$$
2. d is linear, namely
$$d(f + g) = \sum_{i=1}^n D_i(f + g) \, dx_i = df + dg$$
3. d satisfies the product rule, namely
$$d(fg) = (df)g + f(dg)$$

Examples

- We have
$$d(yz \, dx) = z \, dy \wedge dx + y \, dz \wedge dx$$
- We have
$$d(x \, dx) = dx \wedge dx = 0$$

- We have
$$d(x\,dx \wedge dz + z^2\,dx \wedge dy) = 2z\,dz \wedge dx \wedge dy$$
- We have
$$d(f(x, y, z)\,dx \wedge dy \wedge dz) = 0$$

Proposition. *d satisfies the following properties.*[14]

1. *$d : \Omega^k(U) \to \Omega^{k+1}(U)$ is linear.*
2. *d satisfies the product rule, namely*
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^a \omega \wedge d\eta$$
where ω is a differential a-form.
3. *The composition $d \circ d : \Omega^k(U) \to \Omega^{k+2}(U)$ is identically zero.*

Proof. The first property is easy and left for you to check. To see the second, write
$$\omega = \sum_J f_J\,dx_J$$
$$\eta = \sum_J g_J\,dx_J$$
Then
$$\begin{aligned}\omega \wedge \eta &= \left(\sum_J f_J\,dx_J\right) \wedge \left(\sum_J g_J\,dx_J\right) \\ &= \sum_{J,L} f_J g_L\,dx_J \wedge dx_L\end{aligned}$$
Now take the exterior derivative for
$$\begin{aligned}d(\omega \wedge \eta) &= \sum_{J,L} \sum_{i=1}^n D_i(f_J g_L)\,dx_i \wedge dx_J \wedge dx_L \\ &= \sum_{I,J} \sum_{i=1}^n [(D_i f_J) g_L + f_J(D_i g_L)]\,dx_i \wedge dx_J \wedge dx_L \\ &= \left(\sum_J \sum_{i=1}^n D_i f_J\,dx_i \wedge dx_J\right) \wedge \left(\sum_L g_L\,dx_L\right) \\ &\quad + (-1)^a \left(\sum_J f_J\,dx_J\right) \wedge \left(\sum_L \sum_{i=1}^n D_i g_L\,dx_i \wedge dx_L\right)\end{aligned}$$
The negative signs are introduced since moving $dx_i \wedge dx_J \wedge dx_L$ to $dx_J \wedge dx_i \wedge dx_L$ takes precisely a swaps, since we must move the dx_i term passed a terms in dx_J.

[14] d is in fact uniquely characterized by these properties, along with our definition of df for a smooth function $f \in \Omega^0(U)$.

To show the third property, observe that we have

$$\begin{aligned} d(d\omega) &= d\bigg(\sum_J \sum_{i=1}^n D_i f_J \, dx_i \wedge dx_J \bigg) \\ &= \sum_J \sum_{i,j=1}^n D_j D_i f_J \, dx_j \wedge dx_i \wedge dx_J \\ &= \sum_J \sum_{1 \le i < j \le n} \Big(D_j D_i f_J - D_i D_j f_J \Big) \, dx_j \wedge dx_i \wedge dx_J \\ &= 0 \end{aligned}$$

The negative sign is introduced by swapping $dx_i \wedge dx_j$ to $dx_j \wedge dx_i$, and the second order partials are equal by Claurault's theorem (on the midterm!). □

Definition. *A differential form $\omega \in \Omega^k(U)$ is **closed** if $d\omega = 0$.*

Definition. *A differential form $\omega \in \Omega^k(U)$ is **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$.*

Examples

- Consider the case of differential 1-forms on $\mathbb{R}^2$. Let $\omega \in \Omega^1(\mathbb{R}^2)$, and write

  $$\omega = f \, dx + g \, dy$$

  Then we have

  $$d\omega = D_2 f \, dy \wedge dx + D_1 g \, dx \wedge dy = (D_1 g - D_2 f) \, dx \wedge dy$$

  Hence ω is closed precisely when f and g satisfy the relationship $D_1 g = D_2 f$.

  Suppose ω is exact. Then there exists some smooth $h : \mathbb{R}^2 \to \mathbb{R}$ such that

  $$f \, dx + g \, dy = dh = D_1 h \, dx + D_2 h \, dy$$

  ω is exact precisely when $f = D_1 h$ and $g = D_2 h$ for some smooth $h : \mathbb{R}^2 \to \mathbb{R}$.

One way to understand Stokes' theorem is that there is another way to compute the integral of an exact form, namely

$$\int_c \omega = \int_c d\eta = \int_{\partial c} \eta$$

We saw this for 1-forms last time. If $\omega = dh$ is a 1-form and $c : I \to \mathbb{R}^n$ is a 1-cube, then

$$\int_c \omega = h(c(1)) - h(c(0))$$

Note that if $\omega \in \Omega^k(U)$ is exact, then it is also closed, since

$$d\omega = d(d\eta) = 0$$

On the homework, you will show that every closed 1-form $\omega \in \Omega^1(\mathbb{R}^2)$ is exact. This is a special case of *Poincaré's lemma*, which says that in general closed forms on $\mathbb{R}^n$ are exact. This is a fact special to $\mathbb{R}^n$, however.

Examples

- The 1-form $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ given by
$$\omega(x,y) = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
is closed, but not exact.
- The 1-form $dx \in \Omega^1(\mathbb{R}^2)$ is exact. It is the exterior derivative of the function $f(x,y) = x$.

Differential forms detect nontrivial topology of subsets of $\mathbb{R}^n$. For example, the subset $\mathbb{R}^2 \setminus \{0\}$ is not the 'same' topologically as $\mathbb{R}^2$ (meaning they are not homeomorphic or diffeomorphic). You can see this because a loop that travels around the hole in $\mathbb{R}^2 \setminus \{0\}$ cannot be deformed continuously to $\mathbb{R}^2$. These ideas are made precise with *de Rham cohomology*.[15]

Pullbacks

Definition. *Given open subsets $V \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$, a differential form $\omega \in \Omega^k(U)$, and a smooth map $g : V \to U$. The **pullback** of ω along g is the k-form denoted $g^*\omega \in \Omega^k(V)$. It is defined*

$$(g^*\omega)(q)(u_1,\ldots,u_k) = \omega(g(q))\big(Dg(q)(u_1),\ldots,Dg(q)(u_k)\big)$$

The pullback is a way to take forms on the codomain and transfer them to forms on the domain. Note that when $k=0$, there are no vectors involved, so the pullback is just given by precomposition. Namely if $f \in \Omega^0(U)$ is a 0-form then $g^*f = f \circ g$.

Proposition. *The pullback satisfies the following properties.*

1. *$g^* : \Omega^k(U) \to \Omega^k(V)$ is linear*
2. *If $n = m$, then*
$$g^*(dx_1 \wedge \ldots \wedge dx_n) = \det Dg\, dx_1 \wedge \ldots \wedge dx_n$$
3. *g respects wedge products, namely*
$$g^*(\omega \wedge \eta) = g^*\omega \wedge g^*\eta$$
4. *g commutes with the exterior derivative, namely*
$$g^*(d\omega) = d(g^*\omega)$$

[15] Come speak to me or take Math 132 if you're interested!

Examples

- Let $\omega \in \Omega^2(\mathbb{R}^2)$ be given by
$$\omega = dx \wedge dy$$
Define the function
$$\begin{aligned} f : (0,\infty) \times (0,2\pi) &\to \mathbb{R}^2 \\ (r,\theta) &\mapsto (r\cos\theta, r\sin\theta) \end{aligned}$$
Using the above properties, we have
$$\begin{aligned} f^*\omega &= f^*(dx \wedge dy) \\ &= f^*(dx) \wedge f^*(dy) \\ &= d(f^*x) \wedge d(f^*y) \\ &= d(x \circ f) \wedge d(y \circ f) \\ &= d(r\cos\theta) \wedge d(r\sin\theta) \\ &= \big(\cos\theta\, dr + r(-\sin\theta)\, d\theta\big) \wedge \big(\sin\theta\, dr + r\cos\theta\, d\theta\big) \\ &= r\, dr \wedge d\theta \end{aligned}$$
This explains
$$\int\int f(x,y)\, dx \wedge dy = \int\int f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$
When we perform a change of coordinates, we are actually taking a pullback, and the pullback of $dx \wedge dy$ is $r\, dr \wedge d\theta$.

4/22/2019 - Chains and boundaries

Integration and pullbacks

We defined the pullback last class. Usually, this construction is used in the following setting: Let $c : I^k \to \mathbb{R}^n$ be a k-cube and $\omega \in \Omega^k(\mathbb{R}^n)$. The pullback $c^*\omega \in \Omega^k(I^k)$ is given by

$$(c^*\omega)(t)(u_1, \ldots, u_k) = \omega(c(t))\big(Dc(t)(u_1), \ldots, Dc(t)(u_k)\big)$$

Note that $c^*\omega$ is a map

$$c^*\omega : I^k \to \Lambda^k(\mathbb{R}^k) = \operatorname{span}(dt_1 \wedge \ldots \wedge dt_k) \simeq \mathbb{R}$$

Hence we can write $c^*\omega = f\, dt_1 \wedge \ldots \wedge dt_k$ for some smooth $f : I^k \to \mathbb{R}$. To determine this function f, we can evaluate $c^*\omega = f\, dt_1 \wedge \ldots \wedge dt_k$ on the standard basis $e_1, \ldots, e_k$ for

$$\begin{aligned} f(t) &= (f\, dt_1 \wedge \ldots \wedge dt_k)(t)(e_1 \wedge \ldots \wedge e_k) \\ &= (c^*\omega)(t)(e_1, \ldots, e_k) \\ &= \omega(c(t))\big(Dc(t)(e_1), \ldots, Dc(t)(e_k)\big) \\ &= \omega(c(t))(Dc(t)) \end{aligned}$$

since $Dc(t)(e_i)$ is just the ith column of the matrix $Dc(t)$. This should look familiar, as we defined

$$\int_c \omega = \int_{t \in I^k} \omega(c(t))(Dc(t))$$

If we write $c^*\omega = f\, dt_1 \wedge \ldots dt_k$, then

$$\int_c \omega = \int_{I^k} f$$

Examples

- This fact is useful for computation. Let's revisit a previous calculation from this new perspective. Let ω be the winding number form

  $$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

  and let $c(t) = (\cos t, \sin t)$ be the curve that winds around the origin once, defined on $[0, 2\pi]$. By the properties of the pullback from last time, we have

  $$\begin{aligned} c^*\omega &= \frac{-\sin t}{(\cos t)^2 + (\sin t)^2}\ \underbrace{d(\cos t)}_{(-\sin t)\, dt} + \frac{\cos t}{(\cos t)^2 + (\sin t)^2}\ \underbrace{d(\sin t)}_{\cos t\, dt} \\ &= dt \end{aligned}$$

  Hence

  $$\int_c \omega = \int_0^{2\pi} 1\, dt = 2\pi$$

  From a geometric perspective, the vector field associated with ω is always 'parallel' to the curve and therefore yields dot product 1.

- Let

$$
\begin{aligned}
c : [0,1] \times [0,2\pi] &\to \mathbb{R}^2 \\
(r,\theta) &\mapsto (r\cos\theta, r\sin\theta)
\end{aligned}
$$

We computed the pullback of $dx \wedge dy$ last time to be $r\,dr \wedge d\theta$, so we have

$$
\begin{aligned}
\int_c dx \wedge dx &= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r\,dr\,d\theta \\
&= \pi
\end{aligned}
$$

which is the area of the unit circle parameterized by the 2-cube c.

Chains and boundaries

Definition. *A (singular) k-cube in $\mathbb{R}^n$ is a smooth map $c : I^k \to \mathbb{R}^n$.*

Examples

- A point is a 0-cube.
- A curve, a circle, and a knot are all 1-cubes.
- A disc, a square, a sphere, and a torus are all 2-cubes.
- The ***standard k-cube*** in $\mathbb{R}^n$ is given by the inclusion

$$
i : I^k = [0,1]^k \hookrightarrow \mathbb{R}^n
$$

- There is no condition that these maps be injective, so the constant map $c(t) = p$ is a k-cube.
- The map $c : I^2 \to \mathbb{R}^3$ that parameterizes the sphere is a k-cube. It is defined

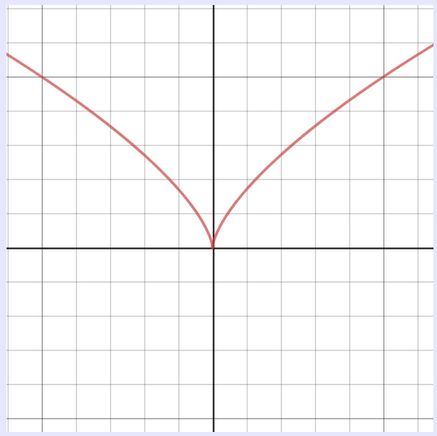
$$
c(\theta,\phi) = \big(\sin(\pi\phi)\cos(2\pi\theta), \sin(\pi\phi)\sin(2\pi\theta), \cos(\pi\phi)\big)
$$

using spherical coordinates.

- The map

$$
\begin{aligned}
c : [0,1] &\to \mathbb{R}^2 \\
t &\mapsto ((2t-1)^3, (2t-1)^2)
\end{aligned}
$$

is a 1-cube called the *cuspidal cubic*.



c is smooth, as its coordinates are given by polynomials. This illustrates that the image of a k-cube need not be a (smooth) manifold. This explains the term *singular*.

- $c(t) = (t, t)$ and $c(t^2, t^2)$ defined on $[0, 1]$ are two distinct 1-cubes in $\mathbb{R}^2$, even though they have the same image.

Two k-cubes are considered equal when they are equal as *maps*, not just if they have the same image.

Given a k-cube $c : I^k \to \mathbb{R}^n$, we can restrict to a face of the cube I^k to yield a $(k-1)$-cube. The boundary map records these restrictions.

Definition. *Let $c : I^k \to \mathbb{R}^n$ be a a k-cube in $\mathbb{R}^n$. For $i = 1, \ldots, k$, define*

$$c_{(i,0)}(t_1, \ldots, t_{k-1}) = (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1})$$
$$c_{(i,1)}(t_1, \ldots, t_{k-1}) = c(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_{k-1})$$

These are $(k-1)$-cubes.

To obtain $c_{(i,\alpha)}$, insert α in the ith position of c. A given k-cube has 2^k possible faces to which we can restrict. These are combined in an object called the *boundary* of c.

Definition. *Let $c : I^k \to \mathbb{R}^n$ be a k-cube. The* ***boundary*** *of c is given by*

$$\partial c = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$

We regard ∂c as a formal linear combination of $(k-1)$-cubes. This just means that we aren't adding these maps $c_{(i,\alpha)}$ pointwise (we are *not* viewing ∂c as a map). We can make this more precise with the following definition.

Definition. *Let $C_k(\mathbb{R}^n)$ be the infinite-dimensional vector space with a basis given by all of the k-cubes on $\mathbb{R}^n$. A vector in $C_k(\mathbb{R}^n)$ is a formal, finite sum of k-cubes with coefficients in $\mathbb{R}$. A* ***k-chain*** *is an element of $C_k(\mathbb{R}^n)$.*

With this language, the boundary

$$\partial : C_k(\mathbb{R}^n) \to C_{k-1}(\mathbb{R}^n)$$

is a linear map from the k-chains to the $(k-1)$-chains. To justify this, note that we have defined ∂ on the basis for $C_k(\mathbb{R}^n)$, so we can just extend ∂ linearly to the entire vector space $C_k(\mathbb{R}^n)$.

Examples

- For our parameterization of the sphere from above, we have

$$\begin{aligned} c_{(1,0)}(t) &= (\sin(\pi t), 0, \cos(\pi t)) \\ c_{(1,1)}(t) &= (\sin(\pi t), 0, \cos(\pi t) \\ c_{(2,0)}(t) &= (0, 0, 1) \\ c_{(2,1)}(t) &= (0, 0, -1) \end{aligned}$$

  Then

$$\partial c = [(0,0,1)] - [(0,0,-1)]$$

  where $[(0,0,a)]$ is the constant cube.

The signs on the formal linear combination in the definition of the boundary are designed to ensure the following result.

Proposition. *For any k-cube c, we have*

$$\partial(\partial c)) = 0$$

Proof. The proof follows directly from the definitions. The main idea is that

$$\begin{aligned} \partial(\partial c)) &= \partial\left(\sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)} \right) \\ &= \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \sum_{j=1}^{k-1} \sum_{\beta=0,1} (-1)^{j+\beta} (c_{(i,\alpha)})_{(j,\beta)} \end{aligned}$$

by the linearity of ∂. The key observation is that each of these terms appears twice, with opposite signs. The $(k-2)$-cubes in the sum have form

$$(c_{(i,\alpha)})_{(j,\beta)} : (t_1, \dots, t_{k-2}) \mapsto c(t_1, \dots, t_{i-1}, \alpha, t_i, \dots, t_{j-1}, \beta, t_j, \dots, t_{k-2})$$

This map appears twice is the resulting sum, as $(c_{(i,\alpha)})_{(j,\beta)}$ and $(c_{(j+1,\beta)})_{(i,\alpha)}$. Then the signs of these two terms will be $(-1)^{i+j+\alpha+\beta}$ and $(-1)^{i+j+\alpha+\beta+1}$, respectively. Intuitively, this happens because there are two possible orders in which we can restrict to an edge of I^k (as there are two faces adjacent to every edge of a cube). $\square$

Stokes' theorem

We can now finally make sense of Stokes' theorem.

Theorem (Stokes' theorem)**.** *Let $\eta \in \Omega^k(\mathbb{R}^n)$ and $c : I^k \to \mathbb{R}^n$ be a k-cube. If $\eta = d\omega$ for some $\omega \in \Omega^{k-1}(\mathbb{R}^n)$, then*

$$\int_c \eta = \int_{\partial c} \omega$$

Note that ∂c is some linear combination of k-cubes:

$$\partial c = \sum_{i=1}^{\ell} a_i d_i$$

Then we are just extending the integral linearly and defining

$$\int_{\partial c} \omega = \int_{\sum_{i=1}^{\ell} a_i d_i} \omega = \sum_{i=1}^{\ell} \int_{d_i} \omega$$

Examples

- Consider the case when $k = n = 1$. Let $\eta \in \Omega^1(\mathbb{R})$, and let $c : I \to \mathbb{R}$ be the standard 1-cube. You'll show on the homework that there exists some smooth function $g \in \Omega^0(\mathbb{R})$ such that $\eta = dg$. Then Stokes' theorem says

  $$\int_c \eta = \int_{\partial c} g$$

  We know $\eta = f\,dx$ for some f, and $dg = g'\,dx$, so $f = g'$. Then we have

  $$\begin{aligned}\int_c \eta &= \int_0^1 f \\ &= \int_0^1 g' \\ &= \int_{\partial c} g \\ &= g(1) - g(0)\end{aligned}$$

  which is just the fundamental theorem of calculus.

- Let $\eta = dx\,dy$ be the area form on $\mathbb{R}^2$, and let $c : I^2 \to \mathbb{R}^2$ be the standard 2-cube. Note that $\eta = d\omega$ for $\omega = x\,dy$. Stokes' theorem yields

  $$\int_c dx\,dy = \int_{\partial c} x\,dy$$

  The left-hand side is

  $$\int_c dx\,dy = \int_{I^2} 1 = 1$$

The right-hand side is

$$\int_{\partial c} \omega = \int_{c_{(1,1)}} \omega - \underbrace{\int_{c_{(1,0)}} \omega + \int_{c_{(2,0)}} \omega - \int_{c_{(2,1)}} \omega}_{0}$$

Each of these terms is a line integral over the vector field given by $F(x, y) = (0, x)$. The last three terms vanish. The middle two trace out paths perpendicular to the vector field F, and F is zero the along the first path.

More generally, this demonstrates how we can use Stokes' theorem to compute areas in $\mathbb{R}^2$ as line integrals along the boundaries. This is *Green's theorem*.

- Let

$$\eta_1 = \frac{x}{x^2 + y^2}\, dx + \frac{y}{x^2 + y^2}\, dy$$
$$\eta_2 = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

be 1-forms defined on $\mathbb{R}^2 \setminus \{0\}$. You'll show on the homework that there exists a smooth function $g \in \Omega^0(\mathbb{R}^2 \setminus \{0\})$ such that $\eta_1 = dg$. Then Stokes' implies

$$\int_c \eta_1 = g(c(1)) - g(c(0))$$

However, there is not such a function g for η_2, so Stokes' theorem says nothing about this case.

4/24/2019 - Stokes' theorem proof

Chains and boundaries

Recall that we defined the vector space of k-chains to be

$$C_k(\mathbb{R}^n) = \left\{ \sum_{i=1}^{m} a_i c_i : a_i \in \mathbb{R}, c_i : I^k \to \mathbb{R}^n \right\}$$

By definition, $C_k(\mathbb{R}^n)$ is spanned linearly independently by k-cubes.

Examples

- Consider the 1-cubes defined by

  $$c_k(t) = \big(k\cos(2\pi t), k\sin(2\pi t)\big)$$

  for $k = 1, 2, 3$. These are concentric circles with radius k. The linear combination

  $$7c_1 - 2c_2 + \sqrt{3}c_3$$

  is a 1-chain on $\mathbb{R}^2$. Note that $c_3(t) - c_2(t) = c_1(t)$ for all $t \in [0,1]$. However, $c_3 - c_2 \neq c_1$ in the vector space $C_1(\mathbb{R}^2)$, because this would result in a nontrivial linear relation $c_1 + c_2 - c_3 = 0$. A sum of k cubes is not a function itself, but rather just a formal expression.

- You will show on homework that there exists a 2-cube $c : I^2 \to \mathbb{R}^2$ such that $\partial c = c_2 - c_1$. The intuition is that the boundary map encodes information about deformations. If we imagine the images of c_1 and c_2 as being stretchy, we can deform c_1 to c_2 by expanding the circle. There exists a family of 1-cubes that interpolate between c_1 and c_2. This family is given by the 2-cube c.

- You will show on the homework that there is no 2-cube $c : I^2 \to \mathbb{R}^2 \setminus \{(1.5, 0)\}$ such that $\partial c = c_2 - c_1$. Again, the intuition is that c_1 cannot be deformed into c_2 in $\mathbb{R}^2 \setminus \{(1.5, 0)\}$.

Lemma. *Fix a 1-cube $c : I \to \mathbb{R}^2$ with $c(0) = c(1)$. Then there exists a 2-cube $d : I^2 \to \mathbb{R}^3$ such that $\partial d = c - [0]$, where $[0](t) = 0$. Intuitively, every loop in $\mathbb{R}^3$ can be deformed to a point.*

Proof. Define the 2-cube $d : I^2 \to \mathbb{R}^3$ by $d(s,t) = sc(t)$. Then

$$\begin{aligned} d(0,t) &= 0 \\ d(1,t) &= c(t) \\ d(s,0) &= c(0) \\ d(s,1) &= c(1) \end{aligned}$$

Since $c(0) = c(1)$ we have $\partial d = c - [0]$. Intuitively, we are interpolating between a point and our loop c. $\square$

The lemma does not hold for any open set of $\mathbb{R}^3$, however. For example, if we replace $\mathbb{R}^3$ with $\mathbb{R}^3 \setminus \{z\text{-axis}\}$, then the loop that wraps once around the z-axis cannot be deformed into a point. In short, we can use cubes and boundaries to probe the topology of a space.[16]

Stokes' theorem

So far, for an open set $U \subset \mathbb{R}^n$ we have defined

- a differential form
$$\omega = \sum_J f_J \, dx_J \in \Omega^k(U)$$
- the exterior derivative
$$d\omega = \sum_J \sum_{i=1}^k D_i f_J \, dx_i \wedge dx_J \in \Omega^{k+1}(U)$$
- a k-cube $c : I^k \to U$, and a k-chain $\sum a_i c_i$
- the boundary
$$\partial c = \sum_{i=1}^k \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$
- the integral over a cube
$$\int_c \omega = \int_{t \in I^k} \omega(c(t))(Dc(t)) \, dt$$
and over a chain
$$\int_{\sum a_i c_i} \omega = \sum a_i \int_{c_i} \omega$$

Theorem (Stokes' theorem)**.** *Let $\omega \in \Omega^{k-1}(U)$ and $z \in C_k(U)$. Then*
$$\int_z d\omega = \int_{\partial z} \omega$$

Proof. To prove the theorem, it suffices to consider the case when $\omega = f_J \, dx_J$ and $z = c : I^k \to U$. Since the derivative, the boundary, and the integral are all linear we can reduce the general case to this form.[17]

[16] This is the idea behind *homology*, which measures the extent to which there are chains with no boundary that do not arise as the boundary of some other chain.

[17] Since we have
$$\int_{\sum a_i c_i} d\Big(\sum_J f_J \, dx_J\Big) = \sum a_i \sum_J \int_{c_i} d(f_J \, dx_J)$$
$$\int_{\partial\left(\sum a_i c_i\right)} \sum_J f_J \, dx_J = \sum a_i \sum_J \int_{\partial c_i} f_J \, dx_J$$

For concreteness, we will consider the case when $U = \mathbb{R}^3$ and $k = 2$. The general case just involves more notation, but the idea is the same. Further suppose $\omega = f\,dx_1 \wedge dx_2$ and $c : I^3 \to \mathbb{R}^2$ is the standard 3-cube.

We'll compute the two sides of the above equation. Note that

$$d\omega = D_3 f\, dx_1 \wedge dx_2 \wedge dx_3$$
$$c^*(d\omega) = d\omega$$

since the pullback along the standard cube is the same form. Then

$$\begin{aligned}
\int_c d\omega &= \int_{I^3} D_3 f f \\
&= \int_{x_1=0}^1 \int_{x_2=0}^1 \int_{x_3=0}^1 D_3 f(x_1, x_2, x_3)\, dx_3\, dx_2\, dx_1 \\
&= \int_{x_1=0}^1 \int_{x_2=0}^1 f(x_1, x_2, 1) - f(x_1, x_2, 0)\, dx_2\, dx_1
\end{aligned}$$

To evaluate the second half of the equation, note we have

$$\partial c = \sum_{i=1}^{3} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$

where

$$\begin{aligned}
c_{(1,\alpha)}(x_1, x_2) &= (\alpha, x_1, x_2) \\
c_{(2,\alpha)}(x_1, x_2) &= (x_1, \alpha, x_2) \\
c_{(3,\alpha)}(x_1, x_2) &= (x_1, x_2, \alpha)
\end{aligned}$$

Then

$$\int_{\partial c} = \sum_{i=1}^{3} \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{c_{(i,\alpha)}} \omega$$

So we want to evaluate these integrals. Since $\omega = f\, dx_1 \wedge dx_2$, we have

$$c^*_{(1,\alpha)}\omega = f \circ c_{(1,\alpha)}\, d\alpha \wedge dx_1 = 0$$
$$\int_{c_{(1,\alpha)}} \omega = 0$$

as $d\alpha = 0$, since α is constant. We similarly have

$$\int_{c_{(2,\alpha)}} \omega = 0$$

However, on the other hand

$$c^*_{(3,\alpha)}\omega = f \circ c_{(3,\alpha)}\, dx_1 \wedge dx_2$$

So therefore

$$\begin{aligned}
\int_{\partial c} \omega &= \int_{c_{(3,1)}} \omega - \int_{c_{(3,0)}} \omega \\
&= \int_{I^2} f(x_1, x_2, 1) - \int_{I^2} f(x_1, x_2, 0)
\end{aligned}$$

which proves that the two sides of the equation are equal. $\square$

The proof is not particularly challenging, but this is because we were careful to define the different constructions properly. It also has many applications.

Vector calculus on $\mathbb{R}^3$

On $\mathbb{R}^3$, we have the correspondence

$$
\begin{tikzcd}
\begin{matrix}\text{0-forms}\\ f\end{matrix} \arrow[r, leftrightarrow] \arrow[d, "d"] & \text{functions } f:\mathbb{R}^3\to\mathbb{R} \arrow[d, "\nabla"] \\
\begin{matrix}\text{1-forms}\\ f_1\,dx_1+f_2\,dx_2+f_3\,dx_3\end{matrix} \arrow[r, leftrightarrow] \arrow[d, "d"] & \text{vector fields } F=(f_1,f_2,f_3):\mathbb{R}^3\to\mathbb{R}^3 \arrow[d, "\text{curl (infinitesimal rotation)}"] \\
\begin{matrix}\text{2-forms}\\ f_1\,dx_2\wedge dx_3+f_2\,dx_3\wedge dx_1+f_3\,dx_1\wedge dx_2\end{matrix} \arrow[r, leftrightarrow] \arrow[d, "d"] & \text{vector fields } F=(f_1,f_2,f_3):\mathbb{R}^3\to\mathbb{R}^3 \arrow[d, "\text{div (infinitesimal expansion)}"] \\
\begin{matrix}\text{3-forms}\\ f\,dx_1\wedge dx_2\wedge dx_3\end{matrix} \arrow[r, leftrightarrow] & \text{functions } f:\mathbb{R}^3\to\mathbb{R}
\end{tikzcd}
$$

There is a version of Stokes' theorem for each of these cases: the fundamental theorem of line integrals, Green's theorem, and the divergence theorem. Stokes' theorem is an abstraction of all of these cases to one theorem.

Applications of Stokes' theorem

We will first examine a fixed point theorem. Let

$$D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$$

denote the closed unit disk and

$$S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$$

be the circle.

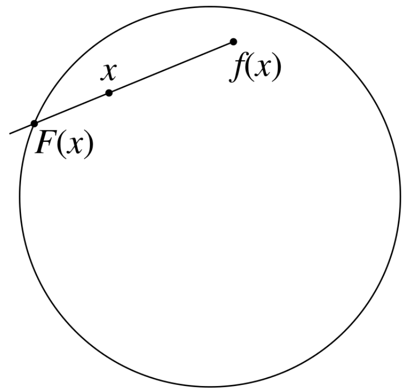
Theorem (Brouwer's fixed point theorem)**.** *Any continuous map $f : D^2 \to D^2$ has a fixed point, namely there exists $x \in D^2$ such that $f(x) = x$.*

The proof follows from the following theorem.

Theorem. *There is no continuous function $g : D^2 \to S^1$ such that $g(x) = x$ for all $x \in S^1$.*

We'll now prove the fixed point theorem using this result.

Proof. Suppose for a contradiction that there exists a map $f : D^2 \to D^2$ that doesn't fix any point. Define the function $g : D^2 \to S^1$ by examining the ray that begins at $f(x)$ and travels through x.



(g is labeled F in the above diagram.) g is continuous, since f is continuous. Since f has no fixed point, this function g is always well-defined. This is a contradiction, so f must have a fixed point. $\square$

We'll now prove the second theorem in the special case in which g is C^1.

Proof. Suppose for contradiction there exists a C^1 function $g : D^2 \to S^1$ such that $g(x) = x$ for all $x \in S^1$. Consider

$$\begin{aligned} i : [0, 2\pi] &\to \mathbb{R}^2 \\ \theta &\mapsto (\cos\theta, \sin\theta) \\ c : [0,1] \times [0, 2\pi] &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta, r\sin\theta) \end{aligned}$$

Note that $\partial c = [0] - i$, where $[0]$ is the constant 1-cube that maps everything to 0. Consider the form $\omega = -y\,dx + x\,dy$. On the one hand,

$$\int_i \omega = 2\pi$$

which is the length of S^1. On the other hand, since g restricted to S^1 is the identity we know $g^*\omega = \omega$ on S^1. Then

$$\begin{aligned} -\int_i \omega &= \int_{[0]} g^*\omega - \int_i g^*\omega = \int_{\partial c} g^*\omega \\ &= \int_c d(g^*\omega) = \int_c g^*(d\omega) \end{aligned}$$

We know $d\omega = 2\,dx \wedge dy$, so

$$g^*(d\omega) = 2\det(Dg)\,dx \wedge dy$$

But $\det(Dg) = 0$ always.[18] Then

$$-\int_i \omega = \int_c g^*(d\omega) = \int_{I^2} 2\det(Dg) = 0$$

which is a contradiction. $\square$

[18]This is because $Dg(a) : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map to a 1-dimensional subspace given by the line tangent to the circle at $g(a)$. Then it cannot be surjective. Email me for more details.

4/29/2019 - Stokes applications: Greens theorem, FTA

Winding numbers

Definition. *A **closed curve** is a 1-cube $c : I \to \mathbb{R}^n$ such that $c(0) = c(1)$.*

Denote

$$A = \mathbb{R}^2 \setminus \{0\}$$
$$B = \mathbb{R}^2 \setminus \{\text{nonnegative } x\text{-axis}\}$$
$$\omega = \frac{1}{x^2+y^2}\Big(- y\,dx + x\,dy\Big) \in \Omega^1(A)$$

We know that

- ω is closed, namely $d\omega = 0$.
- ω is exact on B, with $\omega = d\theta$ where $\theta(x, y)$ is the angle defined by

$$(x, y) = \big(r\cos\theta(x, y), r\sin(\theta(x, y))\big)$$

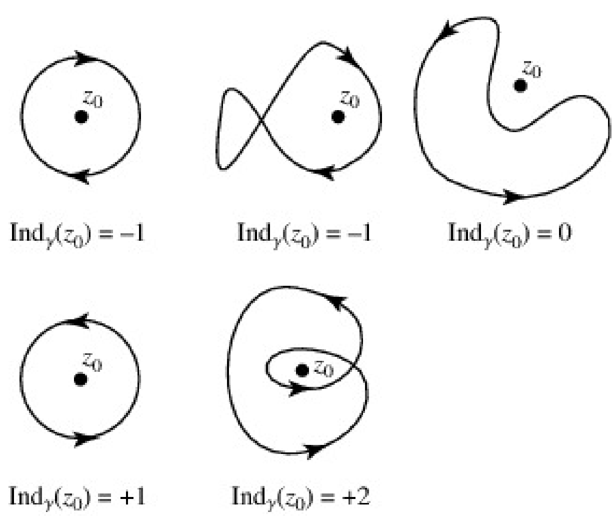
- ω is not exact on A, so there does not exist[a] a function $f : A \to \mathbb{R}$ such that $df = \omega$.

[a] As the integral 'around the circle' is nonzero, even though the endpoints are equal.

Definition. *For a closed curve $c : I \to A$, the **winding number** of c is*

$$\mathit{wind}(c) = \frac{1}{2\pi}\int_c \omega$$

If we view ω as $d\theta$, the winding number measures the net angular change of a closed curve around the origin.



Examples

- For the closed curve $c_{r,n}(t) = (r\cos(2\pi nt), r\sin(2\pi nt))$, we have $\text{wind}(c_{r,n}) = n$. This is because $c_{r,n}$ is just the curve that winds around the origin n times.

Lemma. *For a closed curve c, wind(c) is an integer.*

Proof. The proof rests on two key observations:

1. If c_1, c_2 are two closed curves and there exists a 2-cube $b : I^2 \to A$ such that $\partial b = c_2 - c_1$, then $\text{wind}(c_1) = \text{wind}(c_2)$.

   Recall that the boundary map encodes information about deformations, so the condition $\partial b = c_2 - c_1$ means that it is possible to deform c_1 into c_2 without passing through the origin. This follows from Stokes's theorem, as we have
$$\begin{aligned}\int_{c_2} \omega - \int_{c_1} \omega &= \int_{c_2 - c_1} \omega \\ &= \int_{\partial b} \omega \\ &= \int_b d\omega \\ &= 0\end{aligned}$$
   since ω is closed.

2. For any closed curve c, there exists $n \in \mathbb{Z}$ and a 2-cube b such that $\partial b = c - c_{1,n}$. In other words, every closed curve in A is deformation equivalent to some standard curve that wraps around the origin n times. You'll prove this on the homework.

By these two observations, given any c we can find a 2-cube b and integer n such that $\partial b = c - c_{1,n}$. This implies $\text{wind}(c) = \text{wind}(c_{1,n}) = n$, which is an integer. $\square$

Remark. *If c is a closed form in B, then wind$(c) = 0$.*

This is reasonable, since if we remove the nonnegative x-axis it is impossible for any curve to wind around the origin completely. We can prove this again with Stokes's theorem, as
$$\begin{aligned}\int_c \omega &= \int_c d\theta \\ &= \int_{\partial c} \theta \\ &= \int_{c(1) - c(0)} \theta \\ &= \theta(c(1)) - \theta(c(0))\end{aligned}$$
since $\omega = d\theta$ is exact on B. We previously called this version of Stokes's theorem the fundamental theorem of line integrals.

We see that the fact that ω is closed but not exact is crucial to our definition of the winding number. We can use winding numbers to prove some important results.

The fundamental theorem of algebra

We have the following important theorem.

Theorem. *Let $p = z^n + a_{n-1}z^{n-1} + \ldots + a_0 \in Poly(\mathbb{C})$ be a nonconstant polynomial with complex coefficients. Then p has a root, namely there exists $z \in \mathbb{C}$ such that $p(z) = 0$.*

We used this last semester to prove the existence of eigenvectors for linear operators $T : \mathbb{C}^n \to \mathbb{C}^n$. In this proof

Proof. We'll present a sketch of the proof today. Suppose for contradiction that p has no roots, namely that $p(z) \neq 0$ for all $z \in \mathbb{C}$.

On the one hand, if we choose $r > 0$ to be very small, then $p \circ c_{r,1}$ is a closed curve with image near $p(0)$. This is because the image of the curve $c_{r,1}$ is very close to 0, so the continuity[a] of p implies that the loop $c_{r,1}$ will be sent to a loop closely wrapping around $p(0)$. Then $\text{wind}(p \circ c_{r,1}) = 0$, since this loop doesn't wind around the origin.

On the other hand, if we choose $R > 1$ to be very large, the map p behaves like the highest order term z^n. More precisely, if $|z| > R$, then $p(z) \approx z^n$. Then $p \circ c_{R,1} \approx c_{R^n,n}$, as the image[b] of $c_{R,1}$ under z^n is $c_{R^n,n}$. $p \circ c_{R,1}$ won't be exactly a circle, but it will wrap around the origin n times.[c] Then $\text{wind}(p \circ c_{R,1}) = n$.

However, if d is a 2-cube such that $\partial d = c_{R,1} - c_{r,1}$, then $\partial(p \circ d) = p \circ c_{R,1} - p \circ c_{r,1}$ simply by definition of the boundary. Note that we need $p \circ d$ to be a 2-cube in A, so we are using the assumption that p has no roots. By the lemma

$$n = \text{wind}(p \circ c_{R,1}) = \text{wind}(p \circ c_{r,1}) = 0$$

which is a contradiction. So p has a root. $\square$

[a] We know $p(0)$ is some nonzero point, so take ϵ to be small enough so that the ball of radius ϵ around $p(0)$ doesn't contain 0. Then if we take r to be smaller than the associated δ, we can use the fact that the image of $p \circ c_{r,1}$ is in B to conclude that this closed curve has winding number 0.

[b] We are viewing $\mathbb{C}$ as the plane $\mathbb{R}^2$ under the identification of $a + bi$ with the point (a, b). We see that

$$\begin{aligned} c_{1,1}(t)^n &= (\cos(2\pi t), \sin(2\pi t))^n = \big(\cos(2\pi t) + i\sin(2\pi t)\big)^n = (e^{2\pi i t})^n \\ &= e^{2\pi i n t} = (\cos(2\pi n t), \sin(2\pi n t)) = c_{1,n}(t) \end{aligned}$$

[c] This is the part of the argument that requires some more work.

Previously, given a set $B \subset Q \subset \mathbb{R}^n$, where Q is a closed rectangle, we defined

$$\mathrm{vol}(B) = \int_Q \chi_B$$

We can also use differential forms to describe volumes in $\mathbb{R}^n$.

Proposition. *Fix $B \subset \mathbb{R}^n$, and suppose $c : Q \to \mathbb{R}^n$ is an n-cube such that $c(Q) = B$ and the restriction $c|_{int(Q)}$ is a diffeomorphism onto its image. Then*

$$vol(B) = \pm \int_c dx_1 \wedge \ldots \wedge dx_n$$

Note that the proposition says volume can be computed using *any parameterization.* This follows from the change of variables theorem.

Theorem (Change of variables theorem)**.** *Fix an open set $U \subset \mathbb{R}^n$, and suppose $c : Q \to U$ is an n-cube such that the restriction $c|_{int(Q)}$ is a diffeomorphism onto its image. Then for any $f : U \to \mathbb{R}$, we have*

$$\int_{c(Q)} f = \int_Q (f \circ c) \cdot |\det Dc|$$

The proposition follows quickly from the theorem:

Proof. We'll apply the theorem to the constant function $f = 1$. The left hand side in the above theorem in this case is

$$\int_{c(Q)} 1 = \int \chi_{c(Q)} = \mathrm{vol}(c(Q)) = \mathrm{vol}(B)$$

The right hand side in the above theorem in this case is

$$\int_Q |\det Dc| = \int_c dx_1 \wedge \ldots \wedge dx_n$$

because[19]

$$c^*(dx_1 \wedge \ldots \wedge dx_n) = \det(Dc)\, dx_1 \wedge \ldots \wedge dx_n$$

□

Examples

- We can now revisit volumes of revolution. Fix functions $f, g : [a, b] \to \mathbb{R}$ such that $0 \leq f \leq g$. Then we can consider the set

$$S = \big\{(x, z) : f(x) \leq z \leq g(x) \text{ for } x \in [a, b]\big\}$$

Let B be the three-dimension figure obtained by revolving S about the z-axis. We can

[19] Note that the real function $|\det Dc| : Q \to \mathbb{R}$ is given by the composition

$$Q \xrightarrow{Dc} M_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}$$

parameterize B as

$$
\begin{aligned}
c : [a,b] \times [0,2\pi] \times [0,1] &\to \mathbb{R}^3 \\
(r,\theta,t) &\mapsto \big(r\cos\theta, r\sin\theta, f(r) + t(g(r) - f(r))\big)
\end{aligned}
$$

Compute

$$
Dc = \begin{pmatrix} \cos\theta & -r\sin t & 0 \\ \sin\theta & r\cos\theta & 0 \\ & & g(r) - f(r) \end{pmatrix}
$$

$$
\det Dc = r(g(r) - f(r))
$$

where we are leaving the entries of Dc which are irrelevant when computing the determinant blank for convenience. Then

$$
\begin{aligned}
\mathrm{vol}(B) &= \int_{r=a}^{b} \int_{\theta=0}^{2\pi} \int_{t=0}^{1} r(g(r) - f(r)) \\
&= \int_{r=a}^{b} 2\pi r(g(r) - f(r))
\end{aligned}
$$

as expected.

We have the following classical result:

Theorem. *Let c be a 2-cube, and let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be functions. Then*

$$
\int_{\partial c} f\,dx + g\,dy = \int_c \big(D_1 g - D_2 f\big)\,dx \wedge dy
$$

It should be clear that this is simply an application of Stokes's theorem. We'll focus on the case when

$$
\big(D_1 g - D_2 f\big)\,dx \wedge dy = dx \wedge dy
$$

In this case we are computing the area of some 2-cube c in the plane, as remarked above. Then Green's theorem says that it suffices to take the line integral along the boundary of c of some appropriate form $f\,dx + g\,dy$. It remains to find f and g that satisfy the above conditions on their partial derivatives. One useful such pair of functions is $f(x,y) = -y$ and $g(x,y) = x$.

Examples

- Consider the area bounded by the curves $c_1(\theta) = (\theta\cos\theta, \theta\sin\theta)$ and $c_2(t) = (1-t, \theta)$. We have

$$
\mathrm{vol}(B) = \int_{c_1 + c_2} \frac{1}{2}\big(-y\,dx + x\,dy\big)
$$

Compute

$$
c_1^* \omega = \frac{1}{2}\theta^2\,d\theta
$$

Then we have

$$\mathrm{vol}(B) = \int_0^{2\pi} \frac{1}{2}\theta^2 = \frac{4}{3}\pi^3$$

5/1/2019 - Stokes applications: FTA, planimeters, coconuts

Green's theorem and planimeters

Recall we have the following corollary to Stokes's theorem:

Theorem (Green's theorem)**.** *Let $B \subset \mathbb{R}^2$ be open such that (C) is a closed curve c. Then*

$$\mathit{area}(B) = \int_c x\,dy$$

This follows from Stokes's, since

$$\text{area}(B) = \int_B dx \wedge dy$$

Also note that we can replace $x\,dy$ with any 1-form η that satisfies $d\eta = dx \wedge dy$. There is a physical tool that implements Green's theorem called a *planimeter*.[20]

We'll first consider a different example. Consider a rod moving through the plane. When the rod is moving in a direction orthogonal to its length, it traces out the most area. When it is moving parallel to its length, it doesn't trace out any area. We can use Green's theorem to compute the area traced out by such rod moving through the plane.

Fix a point p on the rod, and let $c : [0, 1] \to \mathbb{R}^2$ be the path traced out by this point. Define $n : [0, 1] \to \mathbb{R}^2$ to be the normal vector to the rod at time t. Infinitesimally, the area swept out by the rod at time t is given by $n(t) \cdot c'(t)$. Then the total area swept out by the rod is

$$A = \int_0^1 n(t) \cdot c'(t)\,dt$$

Lemma. *If the endpoints of the rod trace closed curves c_1, c_2 bound regions B_1 and B_2, then*

$$A = \mathit{area}(B_1) - \mathit{area}(B_2)$$

Proof. This is evident after drawing the paths c_1 and c_2 as well as the areas they bound. □

Corollary. *Given a region B bounded by a closed curve c, if we move the rod such that one end traces out c and the other endpoint traces out a curve that encloses no area, then we can compute (B).*

For example, if one endpoint is fixed and the other endpoint traces out the boundary of B, we have satisfied the conditions of the corollary.

We can now turn to the planimeter.The idea is that we can fix one end of the rod to the circle of a given radius and allow the other end to move freely. Then the endpoint fixed to the circle doesn't trace out any area, so by measuring how much the wheel turns as the other endpoint traces

[20] See the Wikipedia article for how this tool works. The basic idea is that by measuring a turning wheel, we can capture the integral of the form $x\,dy$ along the boundary of some area in the plane.

out an area we can compute the area of a given region.

This idea is closely related to Green's theorem. Define a vector field $n : \mathbb{R}^2 \to \mathbb{R}^2$ given by the unit normal direction to the tracer arm at a given position. Write $n = (n_1, n_2) : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$, and define the 1-form

$$\omega = n_1\,dx + n_2\,dy$$

The claim is that $d\omega = K\,dx \wedge dy$ for some constant K. Then given a region B bounded by a closed curve c, we have

$$\begin{aligned} \text{area}(B) &= \int_B dx \wedge dy \\ &= \frac{1}{K}\int_c \omega \\ &= \int_0^1 n(c(t)) \cdot c'(t)\,dt \end{aligned}$$

which is the total turning of the wheel. The key fact is that the 1-form ω, which is associated to the *physical* setup of our system, actually satisfies $d\omega = K\,dx \wedge dy$.

Fundamental theorem of algebra

We gave a sketch of the proof of the fundamental theorem of algebra, but we'll return to in more detail now.

Theorem (Fundamental theorem of algebra)**.** *Let $p = z^n + a_1 z^{n-1} + \ldots + a_n \in Poly(\mathbb{C})$ be a complex polynomial with degree greater than or equal to 1. Then there exists $z \in \mathbb{C}$ with $p(z) = 0$.*

We define

$$A = \mathbb{R}^2 \setminus \{0\}$$

For a closed curve $c : [0,1] \to A$, we have

$$\text{wind}(c) = \frac{1}{2\pi}\int_c \frac{1}{x^2+y^2}(-y\,dx + x\,dy)$$

Lemma. *We have the following key ideas.*

1. *If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.*
2. *For $q \in Poly_n(\mathbb{C})$, there exists $R > 0$ such that if $|z| > R$, then $|q(z)| > 100$.*
3. *For closed cubes c_1, c_2 in A, if there exists a 2-cube b in A such that $\partial b = c_1 - c_2$, then $wind(c_1) = wind(c_2)$.*

Proof. The first fact follows from basic trigonometric identities (see the footnote from last class). The second fact says that we can get far enough from the origin so that q is at least some constant away from the origin, and it was on the first midterm. The third fact was proved last time, and it follows from Stokes's theorem and the fact that the winding number form is closed. $\square$

We can now prove the theorem.

Proof. The idea is to show that, assuming p has no root, $\text{wind}(p \circ c)$ is both n and z. We will proceed in several steps.

1. There exists $r > 0$ and a 2-cube $b : I^2 \to A$ such that $\partial b = c_{r^n,n} - p \circ c_{r,1}$, where $c_{r,n}(t) = (r\cos(2\pi nt), r\sin(2\pi nt))$. This implies $\text{wind}(p \circ c_{r,1}) = n$.

   To show this, consider the straight-line interpolation

   $$\begin{aligned} b &: I^2 \to A \\ b(s,t) &\mapsto sc_{r^n,n}(t) + (1-s)(p \circ c_{r,1})(t) \end{aligned}$$

   We have the boundary terms

   $$\begin{aligned} b(0,t) &= (p \circ c_{r,1})(t) \\ b(1,t) &= c_{r^n,n}(t) \\ b(s,0) &= s(r^n,0) + (1-s)p(r,0) \\ b(s,1) &= s(r^n,0) + (1-s)p(r,0) \end{aligned}$$

   So

   $$\partial b = p \circ c_{r,1} - c_{r^n,n}$$

   However, there is a problem. b should be a 2-cube in A, and right now we don't know that the image of b is contained in A. If the two points $p \circ c_{r,1}(t)$ and $c_{r^n,n}(t)$ are antipodal, then the interpolation will pass through the origin.

   To fix this, we should take r to be large. Write $z_t = c_{r,1}(t)$, which is the loop of radius r that wraps around once. Then $z_t^n = c_{r^n,n}(t)$. Then we can write

   $$\begin{aligned} b(s,t) &= sz_t^n + (1-s)p(z_t) \\ &= sz_t^n + (1-s)(z_t^n + a_1 z_t^{n-1} + \ldots + a_n) \\ &= z_t^n + (1-s)(a_1 z_t^{n-1} + \ldots + a_n) \end{aligned}$$

   We want to show that $b(s,t)$ is never zero. But this is easy, since $b(s,t)$ is a polynomial evaluated at z_t. Then by the lemma, we can ensure that this is true by taking $|z_t|$ to be large enough. We have $|z_t| = r$, so choose r large enough so that $|b(s,t)| \neq 0$.

   Then $\text{wind}(p \circ c_{r,1}) = \text{wind}(c_{r^n,n}) = n$.

2. Suppose that p has no root, and fix r as in the above step. We'll show $\text{wind}(p \circ c_{r,1}) = 0$. Consider the 2-cube $p \circ d$, where

   $$\begin{aligned} d &: I^2 \to \mathbb{R}^2 \\ (r,t) &\mapsto \big(r\cos(2\pi t), r\sin(2\pi t)\big) \end{aligned}$$

   is the 2-cube that parameterizes the disk. Note that $p \circ d$ is actually a 2-cube in A because we are assuming that p doesn't have any roots. Then

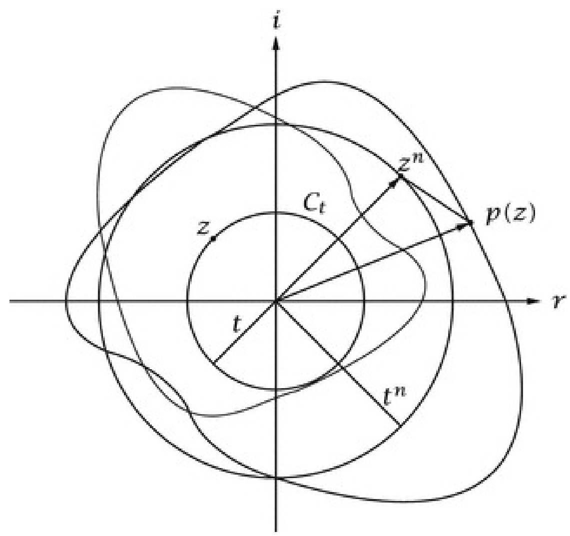
   $$\begin{aligned} (p \circ d)(r,0) &= (p \circ d)(r,1) \\ (p \circ d)(0,t) &= p(0) \\ (p \circ d)(1,t) &= (p \circ c_{r,1})(t) \end{aligned}$$

So

$$\partial(p \circ d) = p \circ c_{r,1} - p(0)$$

This implies that $\text{wind}(p \circ c_{r,1}) = \text{wind}(p(0)) = 0$, which is a contradiction.

$\square$



Thanks for a great year!