Homework 9

Math 25b

Due April 26, 2018

Topics covered: differential forms

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one pdf.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Beckham M.

Problem 1. *Give an example if an example exists. If no example exists, explain why.*

(a) *There exists nonzero* $\omega \in \Omega^1(\mathbb{R}^3)$ *so that* $\omega \wedge d\omega = 0$.

(b) *There exists* $\omega \in \Omega^1(\mathbb{R}^3)$ *so that* $\omega \wedge d\omega(p) \neq 0$ *for all* $p \in \mathbb{R}^3$.

Solution. □

Problem 2. *Fix* $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$*, and view* F *as a vector field. Define forms*

$$\omega_F = F_1\, dx + F_2\, dy + F_3\, dz$$

$$\eta_F = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy.$$

(a) *Define the curl of a vector field by the formula* $d(\omega_F) = \eta_{curl(F)}$. *Give a formula for curl(F).*

(b) *Define the divergence of a vector field by* $d(\eta_F) = div(F)\, dx \wedge dy \wedge dz$. *Give a formula for div(F).*

(c) *Prove that* $curl(\nabla f) = 0$ *for any* $f : \mathbb{R}^3 \to \mathbb{R}$ *and* $div(curl(F)) = 0$ *for any vector field* F. *Hint: for the love of algebra, do. not. compute.*

Solution. □

Problem 3. *Consider the differential forms*

$$\omega = xy\, dx + 3\, dy - yz\, dz \quad \text{and} \quad \eta = x\, dx - yz^2\, dy + 2x\, dz$$

on $\mathbb{R}^3$*, and the function* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $f(x, y, z) = (y, z, x)$. *Verify by direct computation that*

(a) $d(d\omega) = 0$,

(b) $f^*(d\omega) = d(f^*\omega)$, *and*

(c) $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta$.

Solution. □

2 For Davis L.

Problem 4.

(a) *Give an example of a k-form on $\mathbb{R}$ that is not closed.*

(b) *Prove that every closed 1-form on $\mathbb{R}$ is exact. Hint: FTC.*

Solution. □

Problem 5. *Fix a differential 1-form $\omega = f\,dx + g\,dy$ on $\mathbb{R}^2$.*

(a) *Show that if ω is closed, then ω is exact.*[1] *Hint: define $h(x, y)$ as the line integral of ω over the path $c : [0, x + y] \to \mathbb{R}^2$ defined by*

$$c(t) = \begin{cases} (t, 0) & t \in [0, x] \\ (x, t - x) & t \in [x, x + y]. \end{cases}$$

Remark: The path c is not smooth at $t = x$, but don't let that bother you. Hint: you will need a problem from the practice problems for Midterm 2.[2]

(b) *To what extent is h unique? i.e. if $dh_1 = \omega = dh_2$, then what can you conclude about h_1, h_2?*

Solution. □

Problem 6. *Determine which of the following 1-forms $\omega \in \Omega^1(\mathbb{R}^2)$ are exact. If ω is exact, find h so that $dh = \omega$.*

(a) $\omega = x\,dx + y\,dy$

(b) $\omega = xy\,dy$

Solution. □

[1] This is a special case of the *Poincaré lemma.*

[2] It would be good to give a proof of that fact here.

3 For Joey F.

Problem 7. *True or false. Give proof.*

(a) *The kernel of the exterior derivative* $d : \Omega^0(\mathbb{R}^2) \to \Omega^1(\mathbb{R}^2)$ *is finite dimensional.*

(b) *The kernel of the exterior derivative* $d : \Omega^1(\mathbb{R}^2) \to \Omega^2(\mathbb{R}^2)$ *is finite dimensional.*

Solution. □

Problem 8. *Let* $A = \mathbb{R}^2 \setminus \{0\}$. *Consider the 1-form on* A *defined by*

$$\omega = (x\,dx + y\,dy)/(x^2 + y^2).$$

(a) *Show that* ω *is closed.*

(b) *Compute* $\phi^*\omega$, *where* $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$ *is the polar coordinates transformation.*

(c) *Find a function* $h : \mathbb{R}^2 \to \mathbb{R}$ *so that* $\omega = dh$. *Hint: first work in polar coordinates, using (b). Then switch back to* x, y *coordinates.*

Solution. □

Problem 9. *Fix* $r > 0$. *Consider the map* $c : [0,1] \times [0,2\pi] \times [0,\pi] \to \mathbb{R}^3$ *defined by*

$$c(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

Compute $\int_c dx \wedge dy \wedge dz$. *What is the geometric meaning of this computation? Hint: spherical coordinates.*

Solution. □

4 For Laura Z.

Problem 10. *For $R > 0$ and n an integer, define the singular 1-cube $c_{R,n} : [0,1] \to \mathbb{R}^2 \setminus \{0\}$ by*

$$c_{R,n}(t) = (R\cos 2\pi nt, R\sin 2\pi nt).$$

Fix $0 < R_2 < R_1$ and show that there is a singular 2-cube $c : [0,1]^2 \to \mathbb{R}^2 \setminus \{0\}$ such that

$$c_{R_1,n} - c_{R_2,n} = \partial c.$$

Hint: use a "linear interpolation".

Solution. □

Problem 11. *Let $B \subset \mathbb{R}^2$ be the open set*

$$B = \{(x,y) \in \mathbb{R}^2 : \text{ if } x = 0, \text{ then } y < 0\},$$

i.e. B is the complement of the positive x-axis and the origin.

(a) *Observe that for each $(x,y) \in B$, there is a unique $0 < \theta < 2\pi$ and $r \in (0,\infty)$ such that*

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

We use this to define functions $\theta : B \to (0, 2\pi)$ and $r : B \to (0,\infty)$. Show that these functions are C^1 on B.

(b) *Let*

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

Show that $\omega = d\theta$ in B. Hint: observe that "derivatives are local" and that $\tan\theta(x,y) = y/x$ if $x \neq 0$ and $\cot\theta(x,y) = x/y$ if $y \neq 0$.

(c) *Let $g \in \Omega^0(B)$. Show that if $dg = 0$, then g is constant. Hint: MMVT; make sure you use it correctly.*[3]

Solution. □

Problem 12. *Set $A = \mathbb{R}^2 \setminus \{0\}$, and consider $\omega \in \Omega^1(A)$ defined by*

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

(a) *Show ω is closed.*

(b) *Show that ω is not exact on A. Hint: If $\omega = df$ for some $f : A \to \mathbb{R}$, then $f - \theta$ is constant on B. Evaluate the limit of $f(1,y)$ as y approaches 0 through positive and negative values.*

Solution. □

[3] Note that there are examples of $U \subset \mathbb{R}^2$ and $g : U \to \mathbb{R}^2$ so that $Dg(u) = 0$ for all $u \in U$ but g is not constant.