Homework 6

Math 25b

Due March 29, 2018

Topics covered: function convergence, equicontinuity, ODEs

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one pdf.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Laura Z.

Problem 1 (Pugh 2.147)**.** *Let $\mathcal{K}$ be the set of nonempty compact subsets of $\mathbb{R}^2$. For $r > 0$ and $A \in \mathcal{K}$, the r-neighborhood of A is*

$$B_r(A) = \{x \in \mathbb{R}^2 : \exists a \in A \text{ such that } d(x,a) < r\} = \bigcup_{a \in A} B_r(a).$$

For $A, B \in \mathcal{K}$, define

$$D(A,B) = \inf\{r > 0 : A \subset B_r(B) \text{ and } B \subset B_r(A)\}.$$

(a) Show that D is a metric on $\mathcal{K}$.

(b) Let $\mathcal{F} \subset \mathcal{K}$ be the set of finite subsets of $\mathbb{R}^2$. Prove that $\mathcal{F}$ is dense in $\mathcal{K}$.

Solution. □

Problem 2 (Pugh 4.7)**.** *Consider a sequence of continuous functions $f_n : [a,b] \to \mathbb{R}$. The graph G_n of f_n is a compact subset of $\mathbb{R}^2$, i.e. in the notation of the previous problem $G_n \in \mathcal{K}$.*

(a) Prove that $f_n \rightrightarrows f$ if and only if (G_n) converges in $\mathcal{K}$ to the graph G of f.

(b) Formulate equicontinuity in terms of graphs.

Solution. □

Problem 3. *In class we proved that if $f_n : [a,b] \to \mathbb{R}$ are uniformly bounded and equicontinuous, then (f_n) has a uniformly convergent subsequence. Does this statement remain true if we replace $[a,b]$ by $\mathbb{R}$? Explain.*

Solution. □

2 For Beckham M.

Problem 4 (Pugh 4.9)**.** *Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and the sequence $f_n(x) = f(nx)$ is equicontinuous. What can be said about f?*

Solution. □

Problem 5 (Pugh 4.22)**.** *Give an example of a sequence of smooth equicontinuous functions $f_n : [0, 1] \to \mathbb{R}$ whose derivatives are not uniformly bounded.*[1]

Solution. □

Problem 6 (Pugh 4.8)**.** *Is the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by*

$$f_n(x) = \cos(n + x) + \log\left(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)\right)$$

equicontinuous? Prove or disprove. Hint: this problem can be solved without working very hard.

Solution. □

[1]Recall that in class we showed that if $(f_n) \subset C_b$ is a sequence of differentiable functions, and the derivatives are uniformly bounded, then (f_n) is equicontinuous. This exercise shows the converse is false.

3 For Joey F.

Problem 7 (Pugh 4.21)**.** *Suppose that $f_n : [0, 1] \to \mathbb{R}$ are continuous functions and (f_n) is equicontinuous and bounded. Define $f : [0, 1] \to \mathbb{R}$ by*

$$f(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$$

(a) Prove that f is continuous. Remark: note that the supremum of $\{f_n(x) : n \in \mathbb{N}\}$ may either be a limit point or an isolated point of of this set.

(b) Show that (a) fails if (f_n) is not equicontinuous.

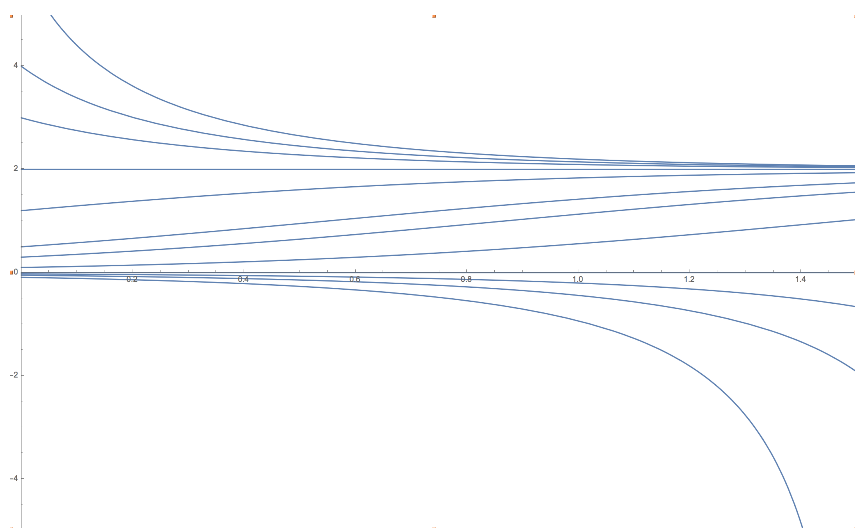
(c) Show that it's possible for f to be continuous, but (f_n) not equicontinuous.

Solution. □

Problem 8. *The logistic equation is the differential equation*

$$x' = ax - bx^2.$$

It models population growth, among other things.[2] *Here is a plot of some solutions for $x' = 2x - x^2$.*



(a) Determine all the constant solutions of $x' = ax - bx^2$. What are the possible initial conditions $x(0) = x_0$? What is the physical interpretation of these solutions?

(b) What happens to solutions at time goes to infinity? Consider finitely many cases, depending on the initial condition. (Determine this by looking at where the function $ax - bx^2$ is positive or negative – what does this tell you about the solutions?) Again, give a physical interpretation, where possible.

[2] It can also be used to study the spread of contagions (diseases, rumor, etc).

Solution. □

Problem 9 (Pugh 4.35). *Consider the ODE $x' = x^2$ on $\mathbb{R}$.*

(a) Find the solution of the ODE with initial condition $x(0) = x_0$. Hint: re-write the ODE as $\frac{x'(t)}{x(t)^2} = 1$; to find an expression for the solution, consider the integral equation

$$\int_0^t \frac{x'(t)}{x(t)^2}\, dt = \int_0^t 1\, dt;$$

the left-hand side can be computed using u-substitution.

(b) Are the solutions to this ODE defined for all time or do they escape to infinite in finite time?

Solution. □

4 For Davis L.

Let $f(t,x)$ be a bounded continuous function $|f(t,x)| \leq M$ defined on $[0,1] \times (-\infty, \infty)$. In this problem you fill in some of the details of the proof of Peano's theorem, which says that the initial-value problem

$$x' = f(t,x), \quad x(0) = x_0$$

has a solution $\phi : [0,1] \to \mathbb{R}$.

Let ϕ_n be the Euler approximation with step size $1/n$. Denoting $t_i = i/n$ for $i = 0, \ldots, n$, recall that ϕ_n is the continuous function with the property that $\phi_n'(t) = f(t_i, \phi_n(t_i))$ for $t \in (t_i, t_{i+1})$. As in our proof, define

$$\Delta_n(t) = \begin{cases} \phi_n'(t) - f(t, \phi_n(t)) & t \neq t_i \\ 0 & t = t_i \end{cases}$$

By the fundamental theorem of calculus,

$$\phi_n(t) = x_0 + \int_0^t \left[f(s, \phi_n(s)) + \Delta_n(s) \right] ds.$$

Problem 10 (Rudin 7.25). *Find a uniformly convergent subsequence of (ϕ_n) as follows.*

(a) Show that $||\Delta_n|| \leq 2M$ and Δ_n is integrable.

(b) Denoting $M_1 = |x_0| + M$, show $||\phi_n|| \leq M_1$ for all n.

(c) Conclude that (ϕ_n) is bounded and equicontinuous, and that after replacing (ϕ_n) by a subsequence, there is a continuous function $\phi : [0,1] \to \mathbb{R}$ so that $\phi_n \rightrightarrows \phi$.

Solution. □

Problem 11. *Assume that $\phi_n \rightrightarrows \phi$ as in the previous problem.*

(a) Prove that $f(t, \phi_n(t)) \to f(t, \phi(t))$ uniformly on $[0,1]$, i.e. show that given $\epsilon > 0$, there exists N so that $n > N$ implies $|f(t, \phi(t)) - f(t, \phi_n(t))| < \epsilon$ for all $t \in [0,1]$. Hint: use that f is uniformly continuous on any closed rectangle (which closed rectangle should you choose?).

(b) Prove that $\Delta_n(t) \to 0$ uniformly on $[0,1]$.

Solution. □

Problem 12. *Conclude that $\phi(t) = x_0 + \int_0^t f(s, \phi(s))\, ds$ for all $t \in [0,1]$ and that ϕ solves the initial-value problem.*

Solution. □