Homework 5

Math 25b

Due March 15, 2018

Topics covered: integrability, measure, Cavalieri, Fubini, function spaces

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one pdf.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Beckham M.

Problem 1 (Pugh 3.39). *Consider the characteristic functions f, g of the intervals $[1,4]$ and $[2,5]$, respectively. The derivatives f' and g' exist almost everywhere. The integration by parts formula says that*

$$\int_0^3 f(x)g'(x)\,dx = f(3)g(3) - f(0)g(0) - \int_0^3 f'(x)g(x)\,dx.$$

But both integrals are zero, while $f(3)g(3) - f(0)g(0) = 1$. Where is the error?

Solution. □

Problem 2 (Pugh 3.31). *Define a Cantor set by removing from $[0,1]$ the middle interval of length $1/4$. From the remaining two intervals F^1, remove the middle intervals of length $1/16$. From the remaining four intervals F^2, remove the middle intervals of length $1/64$, and so on. At the n-th step in the construction F^n consists of 2^n subintervals of F^{n-1}. It is referred to as a "fat Cantor set". Prove that $F = \bigcap F^n$ is not a zero set.* [1] *Hint: it is not enough to add up the lengths of the intervals that are removed; compare with HW4#4.*

Solution. □

Problem 3 (Pugh 3.33). *Let C be the middle-thirds Cantor set and F the fat Cantor set. Prove that χ_C is integrable but χ_F is not.*[2] *Hint: What is the interior of F?*

Solution. □

[1] This shows that being a zero set is not a topological property: F and the usual Cantor set C are homeomorphic, but one is a zero set and the other is not.

[2] There exists a continuous bijection (homeomorphism) $h : [0,1] \to [0,1]$ that sends C to F, so that $\chi_F = h \circ \chi_C$. This exercise shows that compositions of Riemann integrable functions need not be Riemann integrable.

2 For Davis L.

Problem 4. *Let* $f : [a, b] \to \mathbb{R}$ *be bounded, integrable, and non-negative. Let* $A = \{(x, y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$. *Show that* A *is rectifiable and has area* $\int_a^b f$. *Hint: most of the work goes toward showing that* A *is rectifiable. Warning:* f *is not assumed to be continuous!*

Solution. □

Problem 5. *Let* A *and* B *be rectifiable subsets of* $\mathbb{R}^3$. *Let* $A_c = \{(x, y) : (x, y, c) \in A\}$ *and define* B_c *similarly. Suppose* A_c *and* B_c *are rectifiable and have the same area for each* c. *Show that* A *and* B *have the same volume. This is Cavalieri's principle.* [3]

Solution. □

Problem 6.

(a) *Use Fubini's theorem to derive the volume of a cone* C *with base* r *and height* h.

(b) *Fix* $a \geq 0$, *and let* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be continuous functions such that* $f(z) \leq g(z)$ *for each* $z \in [a, b]$. *Consider* $S = \{(y, z) : f(z) \leq y \leq g(z) \text{ and } a \leq z \leq b\}$. *Derive an expression for the volume of a set* $C \subset \mathbb{R}^3$ *obtained by revolving* S *about the* z*-axis.*

(c) *Repeat (b) but now with* f, g *functions of* y, *i.e.* $S = \{(y, z) : a \leq y \leq b \text{ and } f(y) \leq z \leq g(y)\}$. *(Again revolving* S *around the* z*-axis.)* [4]

Solution. □

[3] Look up the "napkin-ring problem," which is a popular application of Cavalieri's principle. Explain it to your friends.

[4] In multivariable calculus, these two methods of computing volumes of revolution are typically called the "washer" and the "shell" methods.

3 For Joey F.

Problem 7. *This problem is about integrating certain unbounded functions.*

(a) *If* $a > 0$, *find* $\lim_{\epsilon\to 0^+} \int_\epsilon^a \frac{1}{\sqrt{x}}\,dx$. *This limit is denoted by* $\int_0^a \frac{1}{\sqrt{x}}\,dx$, *even though the function* $f(x) = 1/\sqrt{x}$ *is not bounded on* $[0,a]$, *no matter how we define* $f(0)$.

(b) *Find* $\int_0^a x^r\,dx$ *if* $-1 < r < 0$.

(c) *Show that* $\int_0^a \frac{1}{x}\,dx$ *does not make sense, even as a limit.*

Solution. □

Problem 8 (Pugh 4.5)**.** *Let* $(f_n) \subset C_b$ *be a sequence of bounded functions on* $[0,1]$. *Prove or disprove: If* (f_n) *converges uniformly to* $f \in C_b$ *and each* f_n *has finitely many discontinuities, then* f *has finitely many discontinuities.*

Solution. □

Problem 9. *Each polynomial* $p \in Poly(\mathbb{R})$ *defines a bounded function on* $[0,1]$. *In this way we view* $Poly(\mathbb{R})$ *as a subspace of* $C_b([0,1],\mathbb{R})$. *Prove that* $Poly(\mathbb{R})$ *is not a closed subspace. Hint: use Taylor polynomials.*

Solution. □

4 For Laura Z.

Problem 10. *This conclusion of this problem is called Darboux's theorem.*

(a) *Suppose f is differentiable on $[a, b]$. Prove that if the minimum of f on $[a, b]$ is at a, then $f'(a) \geq 0$. What is the right conclusion if the minimum is at b?*

(b) *Suppose $f'(a) < 0$ and $f'(b) > 0$. Show that $f'(x) = 0$ for some $x \in (a, b)$. Hint: the intermediate value theorem does not apply here (why?).*

(c) *Use (b) to conclude that if $f'(a) < c < f'(b)$, then $f'(x) = c$ for some $x \in (a, b)$.*

Solution. □

Problem 11. *Use the fundamental theorem of calculus and Darboux's theorem to give an alternate proof of the intermediate value theorem.*

Solution. □

Problem 12. *In this problem, you prove a special case of the Riemann–Lebesgue theorem. Let $C \subset \mathbb{R}^n$ be a bounded set, contained in a closed rectangle Q.*

(a) *Suppose that $\int_Q \chi_C$ exists. Prove that $\operatorname{bd} C$ has measure 0. In fact, prove that for every $\epsilon > 0$, there are finitely many rectangles $Q_1, \ldots, Q_k$ that cover C such that $\sum$ vol$(Q_i) < \epsilon$.* [5]

(b) *Suppose that $\operatorname{bd}(C)$ has measure 0. Prove that $\int_Q \chi_C$ exists. Hint: build a partition P of Q with $U(f, P) - L(f, P) < \epsilon$; use that Q is covering compact.*

Solution. □

[5] In this case we say that C has *content 0.*