Homework 2

Math 25b

Due February 15, 2018

Topics covered: sequences, limits, continuity, open/closed sets

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).

1 For Laura Z.

Problem 1. *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map.*

(a) *Show that there is a number* M *such that* $|Th| \leq M|h|$ *for* $h \in \mathbb{R}^n$*. Hint: Estimate* $|Th|$ *in terms of* $|h|$ *and the entries in the matrix of* T*.*

(b) *Show that* T *is continuous.*

Solution. □

Problem 2. *Let* (x_n) *be a sequence in* $\mathbb{R}^n$*. Prove that the following statements are equivalent*

(a) *The sequence* (x_n) *converges to* a*.*

(b) *Every subsequence of* (x_n) *has a further subsequence that converges to* a*.*

Solution. □

Problem 3. *In this problem you prove the "Squeeze Theorem." Suppose that* $f(x) \leq g(x) \leq h(x)$ *and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$*. Prove that* $\lim_{x\to a} g(x)$ *exists and*

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

Solution. □

2 For Beckham M.

Problem 4. *In each case, find δ such that $|f(x) - \ell| < \epsilon$ for all x satisfying $0 < |x - a| < \delta$.*

(a) $f(x) = \frac{1}{x}$; $a = 1$, $\ell = 1$.

(b) $f(x) = \sqrt{|x|}$; $a = 0$, $\ell = 0$.

(c) $f(x) = \sqrt{x}$; $a = 1$, $\ell = 1$.

Solution. □

Problem 5.

(a) *Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f = 0$ for all x in a dense set A, then $f(x) = 0$ for all x.*

(b) *Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x+y) = f(x) + f(y)$ for all x, y then f is linear, i.e. there exists c so that $f(x) = cx$ for all x.* [1]

Solution. □

Problem 6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Prove that the following statements are equivalent:*

(a) *For each $x \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $\delta > 0$ so that $|t - x| < \delta$ implies $|f(t) - f(x)| < \epsilon$.*

(b) *For every $U \subset \mathbb{R}^m$ open, the set $f^{-1}(U)$ is open.*

Solution. □

[1] There exist functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x+y) = f(x) + f(y)$ for all x, y, but are not continuous, but they are not easy to construct.

3 For Davis L.

Problem 7 (Pugh 1.42). *A function defined on $[a, b]$ or (a, b) is uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Note that this δ cannot depend on a, it can only depend on ϵ. With ordinary continuity, δ can depend on both a and ϵ.*

(a) Show that any uniformly continuous function is continuous. Give an example of a function $f : (0, 1) \to \mathbb{R}$ that is continuity but not uniformly continuous.

(b) Is the function $2x$ uniformly continuous on $(-\infty, \infty)$? What about x^2? Give proof.

Solution. □

Problem 8 (Pugh 1.43). *In this exercise you prove that a continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous. Fix $\epsilon > 0$. Consider the sets*

$$A(\delta) = \{u \in [a, b] : \text{ if } x, t \in [a, u] \text{ and } |x - t| < \delta, \text{ then } |f(x) - f(t)| < \epsilon\}$$

and

$$A = \bigcup_{\delta > 0} A(\delta).$$

Use the least upper bound property to prove that $b \in A$. Infer that f is uniformly continuous.[2]

Solution. □

Problem 9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function with coordinate functions $f = (f_1, \ldots, f_m)$. Show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f_i(x) \to b_i$ for each $i = 1, \ldots, m$.*

Solution. □

[2] According to Pugh, the fact that a continuity on $[a, b]$ implies uniform continuity is one of the important, fundamental principles of continuous function.

4 For Joey F.

Problem 10. *Let $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ denote the subset of invertible matrices.*[3] *In this problem you'll show that $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.*

(a) *Show that the determinant* $\det : M_n(\mathbb{R}) \to \mathbb{R}$ *is a continuous function. Hint: recall the formula for the determinant from Math 25a.*[4]

(b) *Conclude that $GL_n(\mathbb{R})$ is open. Hint: recall the open set characterization of continuity.*

Solution. □

Problem 11. *Consider $f : [-\frac{2}{\pi}, \frac{2}{\pi}] \to \mathbb{R}$ defined by* $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$

(a) *Show that f satisfies the conclusion of the Intermediate Value Theorem (i.e. for any $a < b$ in $[-\frac{2}{\pi}, \frac{2}{\pi}]$ and d with $f(a) < d < f(b)$, there exists $c \in [a, b]$ with $f(c) = d$). Does f satisfy the hypothesis of IVT? Hint: you can easily treat the case $[a, b]$ does not contain 0.*

(b) *Suppose that $f : [a, b] \to \mathbb{R}$ satisfies the conclusion of IVT and that f takes on each value only once (i.e. f is injective). Prove that f is continuous. Hint: It might help to show that f is either increasing or decreasing (recall that f is increasing if $x < y$ implies $f(x) < f(y)$).*

Solution. □

Problem 12. *Prove that there does not exist a continuous function on $\mathbb{R}$ that takes on every value exactly twice. Is the same statement true if we replace twice by thrice?*

Solution. □

[3] In fact the set of invertible matrices forms a group, known as the general linear group.

[4] If you weren't here, ask someone.