Homework 10

Math 25b

Due May 1, 2018

Topics covered: Stokes, Stokes, Stokes

Instructions:

- The homework is divided into one part for each CA. You will submit the assignment on Canvas as one pdf.
- If you collaborate with other students, please mention this near the corresponding problems.

1 For Laura Z.

Problem 1. *Fix* $0 < a < b$ *and define* $c(r, \theta) = (r \cos\theta, r \sin\theta)$ *for* $a \leq r \leq b$ *and* $0 \leq \theta \leq 2\pi$. *Put* $\omega = x^3\, dy$ *and compute both*

$$\int_c d\omega \quad \text{and} \quad \int_{\partial c} \omega$$

to verify that they are equal.

Solution. □

Problem 2. *Let* $\omega = f(x)\, dx$ *be a* 1*-form on* $[0, 1]$. *Show that there is a function* $g : [0, 1] \to \mathbb{R}$ *with* $g(0) = g(1)$ *and a unique number* λ *such that* $\omega - \lambda\, dx = dg$. *Hint: integrate* $\omega - \lambda\, dx = dg$ *to find* λ*; then solve for* g.

Solution. □

Problem 3. *Fix* $U \subset \mathbb{R}^n$ *open and* $\omega \in \Omega^k(U)$.

(a) Show that if $\omega \neq 0$ *then there is a* k*-cube* c *such that* $\int_c \omega \neq 0$.

(b) Use (a) to give an alternate proof that $d^2\omega = 0$ *for any* ω. *Hint: use Stokes and* $\partial^2 = 0$.

Solution. □

2 For Beckham M.

Problem 4. *Let c_1, c_2 be 1-cubes in $\mathbb{R}^2$ with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$ (so the images of c_1, c_2 are curves in the plane with the same endpoints).*

(a) *Show that there is a 2-cube c such that $\partial c = c_1 - c_2 + c_3 - c_4$, where c_3 and c_4 are constant 1-cubes.*

(b) *Conclude that $\int_{c_1} \omega = \int_{c_2} \omega$ if ω is exact. Give a counterexample on $\mathbb{R}^2 \setminus 0$ if ω is merely closed.*

(c) *Prove the converse of (b): If ω is a 1-form on $U \subset \mathbb{R}^2$ and $\int_{c_1} \omega = \int_{c_2} \omega$ for all c_1 and c_2 on U with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$, then ω is exact. Hint: fix a basepoint o, and define $g(p) = \int_c \omega$, where c is a path from o to p; explain why g is well-defined, and show $dg = \omega$ use a similar strategy as HW9#5. (This problem is a bit tricky – if you are stuck, ask for help.)*

Solution. □

Problem 5. *Let $A \in M_3(\mathbb{R})$ be a matrix all of whose entries are positive. Use Brouwer's fixed point theorem to prove that A has an eigenvector.*

Solution. □

Problem 6. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map such that $|f(x)| = 1$ for all $x \in \mathbb{R}^n$. Show that $\det Df(x) = 0$ for all $x \in \mathbb{R}^n$.*

Solution. □

3 For Davis L.

Problem 7. *Let $D^3 \subset \mathbb{R}^3$ be the closed unit disk, and denote $S^2 \subset \mathbb{R}^3$ the 2-sphere.*

(a) Show that there is no C^1 map $g : D^3 \to S^2$ so that $g(x) = x$ for all $x \in S^2$. (Your proof should contain more details/justificaiton than the proof for maps $D^2 \to S^1$ given in class.)

(b) The proof given in class for maps $g : D^2 \to S^1$ had a small lie[1]*. Find it, and explain why it doesn't affect the proof.*

Solution. □

Notation. In the remaining problems, we'll use the following notation.

- $A = \mathbb{R}^2 \setminus \{0\}$ and $B = A \setminus X_+$, where X_+ is the positive x-axis.
- $\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$ and $\theta : B \to \mathbb{R}$ is the angle function with $\omega = d\theta$ on B (as in HW9).
- $c_{R,n} : [0,1] \to A$ is the 1-cube defined by $c_{R,n}(t) = (R\, \cos 2\pi nt, R \sin 2\pi nt)$.

Problem 8.

(a) Show that $\int_{\partial z} \omega = 0$ for any 2-chain $z \in C_2(A)$. Hint: Stokes.

(b) Use (a) to show that $c_{R,n} \neq \partial z$ for any 2-chain z on A.

Solution. □

[1] oops! 😇

4 For Joey F.

Problem 9. *Let c is a singular 1-cube in A with $c(0) = c(1)$.*

(a) *Show that $\int_c \omega = 2\pi n$ for some integer n. For simplicity, assume that $c(0) = c(1)$ is on the x-axis and that there are numbers $0 = t_1 < t_2 < \cdots < t_m = 1$ so that $c(t_i) \in X_+$ for each i, and $c(t) \in B$ for $t \notin \{t_1, \ldots, t_m\}$. Hint: use that $\omega = d\theta$ on B and use FTC.*

(b) *Take n as in (a). Find a 2-cube $b : [0,1]^2 \to A$ so that $\partial b = c - c_{1,n}$. Hint: one approach is to define functions $r(t) = |c(t)|$ and $\theta(t) = \int_{c|_{[0,t]}} \omega$ and to do a linear interpolation between $r(t)$ and 1 and between $\theta(t)$ and $2\pi nt$. In any case, you'll need to make sure that the image of b is contained in A.*

Together (a) and (b) show that every loop in A can be deformed to the loop $c_{1,n}$, where n is the winding number of c. This is a basic result about the topology of A, which is crucial to the proof of the fundamental theorem of algebra.

Solution. □

Problem 10. *Show that the integer n of Problem 9 is unique. This integer is called the* <u>winding number</u> *of c around 0. Hint: use Problems 8 and 9.*

Solution. □

Problem 11. *Let η be a closed 1-form on A. In this problem you prove that $\eta = \lambda\,\omega + dg$ for some $\lambda \in \mathbb{R}$ and $g : A \to \mathbb{R}$.* [2]

(a) *Find λ. Hint: se Problem 2 to write $c_{R,1}^*(\eta) = 2\pi\lambda_R\, dt + d(g_R)$ for some λ_R and g_R with $g_R(0) = g_R(1)$, and show that all numbers λ_R have the same value λ (using HW9#10).*

(b) *Show that $\alpha := \eta - \lambda\omega$ is exact. Hint: use Problem 4. To show $\int_{c_1} \alpha = \int_{c_2} \alpha$, show that there exists n and a 2-cube c so that $c_2 - c_1 = c_{1,n} + \partial c$; this is similar to Problem 9(b).*

Solution. □

[2] As a consequence, after you finish this problem, you'll have computed the 1-*dimensional de Rham cohomology vector space of* A, often denoted $H^1_{dR}(A)$. This vector space is defined as the quotient of the vector space of closed 1-forms by the subspace of exact 1-forms. This problem shows $H^1_{dR}(A) \simeq \mathbb{R}$, spanned by the winding number form. The fact that $H^1_{dR}(A) \neq 0$ is indicative of the fact that A has nontrivial topology. For $k \geq 0$ and $U \subset \mathbb{R}^n$ open, the De Rham cohomology $H^k_{dR}(U)$ (defined similarly as closed k-forms modulo exact k-forms) gives a precise measure of the extent to which the Fundamental Theorem of Calculus fails to extend to k-forms on U. You can learn more about this in Math 132.