Homework 1

Math 25b

Due February 8, 2018 at 5pm

Topics covered: Continuity, least upper bound property, continuity theorems, Dedekind cuts

Instructions:

- The homework is divided into one part for each CA. You will submit each part to Canvas (please separate them when you submit).
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Pugh's book. I've indicated this next to the problems (e.g. "Pugh 1.2" means problem 2 of chapter 1 in Pugh).

1 For Beckham M.

Problem 1 (Pugh 1.13). *Let $b = \sup S$, where S is a bounded nonempty subset of $\mathbb{R}$.*

(a) *Given $\epsilon > 0$ show that there exists $s \in S$ with $b - \epsilon \leq s \leq b$. Can $s \in S$ always be found so that $b - \epsilon < s < b$?*

(b) *If $x \subset \mathbb{Q}$ be a cut. Show that $x = \sup x$ in $\mathbb{R}$.*

Solution. □

Problem 2 (Pugh 1.15). *In this exercise you prove that the function $f(x) = x^n$ is continuous. Fix $a \in \mathbb{R}$, $n \in \mathbb{N}$, and $\epsilon > 0$. Show that for some $\delta > 0$, if $x \in \mathbb{R}$ and $|x - a| < \delta$, then $|x^n - a^n| < \epsilon$. Hint: first do $n = 1$, $n = 2$, and then do induction on n using the identity*

$$(x^n - a^n) = (x - a)(x^{n-1} + x^{n-2}a + \cdots + a^{n-1})$$

Solution. □

Problem 3 (Pugh 1.16). *Here you'll prove the existence and uniqueness of n-th roots in $\mathbb{R}$, using the least upper bound property. Given $x > 0$ and $n \in \mathbb{N}$ prove that there exists unique $y > 0$ such that $y^n = x$. Hint: Consider $y = \sup\{s \in \mathbb{R} : s^n \leq x\}$. Show y^n cannot be $< x$ or $> x$ using the previous exercise.*

Solution. □

2 For Davis L.

Problem 4 (Pugh 1.18)**.** *In this exercise you prove that the real numbers correspond bijectively to decimal expansions not terminating in an infinite string of nines.*[1] *Given $x \in \mathbb{R}$ (with $\mathbb{R}$ defined using cuts), define a decimal expansion $N.x_1x_2\ldots$, where N is the largest integer $\le x$, x_1 is the largest integer $\le 10(x-N)$, x_2 is the largest integer $\le 100(x-(N+x_1/10))$, and so on.*

(a) Show that each x_k is a digit between 0 and 9.

(b) Show that for each k there is $\ell \ge k$ so that $x_\ell \ne 9$.

(c) Conversely, show that for any such expansion $N.x_1x_2\ldots$ not terminating in an infinite string of nines, the set

$$\{N, N + \frac{x_1}{10}, N + \frac{x_1}{10} + \frac{x_2}{100}, \cdots\}$$

is bounded and its least upper bound is a real number x with decimal expansion $N.x_1x_2\ldots$.

Solution. □

Problem 5 (Pugh 1.19)**.** *Formulate the definition of the greatest lower bound* $\inf A$ *of a set of real numbers. State a "greatest lower bound property" for $\mathbb{R}$ and show it is equivalent to the least upper bound property of $\mathbb{R}$.*

Solution. □

Problem 6 (Pugh 1.22)**.** *Let A be a set. We say $a \in A$ is a fixed point of a function $f : A \to A$ if $f(a) = a$. The graph of $f : A \to A$ is the set*

$$G = \{(a, f(a)) : a \in A\} \subset A \times A.$$

The diagonal of $A \times A$ is the subset $\{(a, a) : a \in A\}$.

(a) Show that $f : A \to A$ has a fixed point if and only if the graph of f intersects the diagonal.

(b) Prove that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. Hint: IVT.[2]

(c) Is the same true for discontinuous f? What about continuous functions $f : (0, 1) \to (0, 1)$?

Solution. □

[1] One can prove a similar result for binary expansions or any other base.

[2] This is a special case the Brouwer fixed point theorem, which has surprising consequences like "It's impossible to comb the hairs on a coconut without creating a bald spot."

3 For Joey F.

Problem 7 (Pugh 1.24)**.** *Given a cube in* $\mathbb{R}^n$*, what is the largest ball it contains? Given a ball in* $\mathbb{R}^n$*, what is the largest cube it contains? What is the largest ball and cube contained in a given box in* $\mathbb{R}^n$*?* [3]

Solution. □

Problem 8 (Pugh 1.41)**.** *Let* v *be a value of a continuous function* $f : [a, b] \to \mathbb{R}$ *(i.e.* v *is in the image of* f*). Use the least upper bound property to prove that there is a smallest and largest* $x \in [a, b]$ *so that* $f(x) = v$*. Observe that this property does not hold for the function* $f : (0, 1) \to \mathbb{R}$ *defined by* $f(x) = \sin(1/x)$*.*

Solution. □

Problem 9. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function. We say that* f *approaches* b *as* x *approaches* a *from the right if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that* $|f(x) - b| < \epsilon$ *whenever* $0 < x - a < \delta$*. In this case we write* $\lim_{x\to a+} f(x) = b$*. Similarly, we say that* f *approaches* b *as* x *approaches* a *from the left if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that* $|f(x) - b| < \epsilon$ *whenever* $0 < a - x < \delta$*. Then we write* $\lim_{x\to a-} f(x) = b$*. Show that the following statements are equivalent.*

(i) $\lim_{x\to a} f = b$

(ii) $\lim_{x\to a+} f = b$ *and* $\lim_{x\to a-} f = b$*.*

Solution. □

[3] This exercise will be useful later.

4 For Laura Z.

Problem 10. *Suppose f, g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in [a, b]$. Hint: if your proof is not very short, then they are not the right one.*

Solution. □

Problem 11. *Find the following limits. Hint: it helps to start with an algebraic simplification.*

(a) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x}$

(b) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$

Solution. □

Problem 12. *In this problem, $x, y \in \mathbb{R}$.* [4]

(a) *Suppose $y - x > 1$. Prove that there is an integer k such that $x < k < y$. Hint: let ℓ be the largest integer less than x and consider $\ell + 1$.*

(b) *Suppose $x < y$. Prove that there is a rational number r such that $x < r < y$. Hint: Find $n \in \mathbb{N}$ so that $n(y - x) > 1$.*

(c) *Suppose $r < s$ are rational numbers. Prove that there is an irrational number between r and s. Hint: You may find it useful that $\sqrt{2}$ is irrational.*

(d) *Suppose $x < y$. Prove that there is an irrational number between x and y.*

A subset $A \subset \mathbb{R}$ is called dense if every open interval contains an element of A. The problem above shows that the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$.

Solution. □

[4] For this problem, at several points, you might think a certain statement is "obvious," but it's useful to remember that there are ordered fields that don't satisfy the LUB property, where the set of natural numbers is bounded(!) and where $0 \neq \inf\{\frac{1}{n} : n \in \mathbb{N}\}$ (!).