Homework 3

Math 241

Due October 14, 2019 by 5pm

Topics covered: Covering spaces, path lifting, monodromy, covers of figure-8, covering homotopy, deck transformations, lifting characterization

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.

Problem 1. *Determine if each of the following is a covering space. Give proof (you may freely use facts from complex analysis, etc).*

(a) $p : \mathbb{C}^\times \to \mathbb{C}^\times$ *defined by* $p(z) = z^2$.

(b) $p : \mathbb{C} \to \mathbb{C}$ *defined by* $p(z) = z^2$.

(c) $p : \mathbb{C} \to \mathbb{C}^\times$ *defined by* $p(z) = e^z$.

(d) $p : (0, 2) \to S^1$ *defined by* $p(t) = e^{2\pi i t}$.

Here $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Solution. □

Problem 2 (Hatcher 1.3.4)**.** *Construct a simply-connected covering space of the space* $X \subset \mathbb{R}^3$ *that is the union of a sphere and a diameter. Do the same when* X *is the union of a sphere and a circle intersecting it in two points.*[1] *Make sure you explain why the spaces you construct are simply connected.*

Solution. □

Problem 3 (Hatcher 1.3.5)**.** *Let* X *be the subspace of* $\mathbb{R}^2$ *consisting of the four sides of the square* $[0, 1] \times [0, 1]$ *together with the segments of vertical lines* $x = 1/2, 1/3, 1/4, \ldots$ *inside the square. Show that for every covering space* $\tilde{X} \to X$ *there is some neighborhood of the left edge of* X *that lifts homeomorphically to* $\tilde{X}$*. Deduce that* X *has no simply-connected covering space.*

Solution. □

Problem 4 (Hatcher 1.3.7)**.** *Let* $Y \subset \mathbb{R}^2$ *be the union of the graph of* $\sin(1/x)$ *on* $[0, 1]$*, the segment* $\{0\} \times [-1, 1]$*, and a disjoint arc* α *connecting* $(0, 0)$ *to* $(1/\pi, 0)$ *(see Hatcher for a picture). Collapsing the (closure of the) complement of* $\alpha \subset Y$ *to a point defines a map* $f : Y \to S^1$*. Show that* f *does not lift to the covering space* $p : \mathbb{R} \to S^1$*, even though* $\pi_1(Y) = 0$*. This shows that local connectedness is necessary for the lifting criterion.*

Solution. □

Problem 5. *Let* $p : Y \to X$ *be a covering space. Prove that* $\pi_k(Y) \cong \pi_k(X)$ *for* $k \geq 2$*. Conclude that* $\pi_k(T^n) = 0$ *for* $k \geq 2$*. (Remark: a connected space for which* $\pi_k(X) = 0$ *for* $k \geq 2$ *is called* aspherical*.)*

Solution. □

Problem 6. *Fix a finite cell complex* X*, and let* $p : Y \to X$ *be a finite covering space.*

(a) *Construct a cell structure on* Y *so that* p *sends cells of* Y *to cells of* X*.*

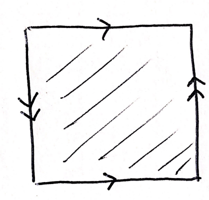
[1] Hint: it may be helpful to first construct a simply-connected cover for the graph that looks like θ.

(b) *Define the Euler characteristic $\chi(X)$ of a cell complex X as the alternating sum $\sum(-1)^i c_i$ of the number of cells in each dimension. Prove that for $Y \to X$ as above $\chi(Y) = d\chi(X)$, where d is the degree of the cover. (Later we will prove that the Euler characteristic is a topological invariant.)*

Solution. □

Problem 7.

(a) *Construct two 2-fold covers of the Klein bottle (the surface pictured below), one by itself and one by the torus.*



(b) *Show that the torus is not a cover of a closed surface of genus 2.*

Solution. □

Problem 8. *Let $p : \mathbb{R} \to S^1$ be the covering space $p(t) = e^{2\pi i t}$. Let $q : Y \to S^1$ be some other covering space. Fix $y_0 \in Y$ so that $q(y_0) = 1 \in S^1$. Show that there exists a covering map $f : \mathbb{R} \to Y$ so that $f(0) = y_0$ and $q \circ f = p$. Do this "by hand", without using the lifting criterion.*[2]

Solution. □

Problem 9 (Hatcher 1.3.8)**.** *Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be simply-connected covering spaces of connected, locally connected spaces X, Y. Show that if $X \simeq Y$, then $\tilde{X} \simeq \tilde{Y}$.* [3]

Solution. □

Problem 10 (Hatcher 1.3.25)**.** *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 \setminus \{0\}$.*

(a) *Show this action is a covering space action. Given $(x, y) \in X$ find an explicit rectangle whose translates under $\mathbb{Z}$ are disjoint.*

(b) *Show that $Y := X/\mathbb{Z}$ is not Hausdorff, and describe it as a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$ (this will not be a disjoint union).* [4]

[2]Hint: It could help to observe/show that there is a constant $\delta > 0$ so that any arc on S^1 of radius $< \delta$ is evenly covered under $q : Y \to S^1$.

[3]Hint: lift maps giving a homotopy equivalence to maps between the covers whose composition is homotopy equivalent to a deck transformation; use the first part of Hatcher Exercise 0.11 (this is easy so you should prove it).

[4]Hint: observe that ϕ preserves the four standard open half-planes (upper, lower, right, left). Determine the quotient of ϕ acting on each of these half-planes.

(c) Compute $\pi_1(Y)$*. Hint: it is an extension* $1 \to \mathbb{Z} \to \pi_1(Y) \to \mathbb{Z} \to 1$ *(why?). How many such extensions are there?*

Solution. □

Problem 11. *Read about* SO(3) *in Bredon Chapter III, Section 10. Use* $\pi_1(\mathrm{SO}(3))$ *to explain why car shades fold oddly.* [5]

Solution. □

[5] Hint: the main physical fact about the car shade that you need is that, if you image a tangent vector along the metal band, when the band is moved, its orthogonal complement does not rotate.