Homework 2

Math 241

Due September 30, 2019 by 5pm

Topics covered: Homotopy groups, fundamental group of circle, van Kampen, change of basepoint, functoriality

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.

Problem 1. *Let G be a topological group[1] with identity $e \in G$. For loops $f, g : (I, \partial I) \to (G, e)$, define a loop $(f \bullet g)(t) := f(t)g(t)$ by pointwise multiplication in G.*

(a) *Show that $f * g \simeq f \bullet g$ rel $*$.*

(b) *Show that $\pi_1(G, e)$ is abelian. Hint: use (a).*

Solution. □

Problem 2 (Hatcher 1.1.6)**.** *We can regard $\pi_1(X, x_0)$ as the set of homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ without conditions on basepoints. There is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. In other words, $[S^1, X]$ is the set of conjugacy classes of elements in $\pi_1(X)$.*

Solution. □

Problem 3.

(a) *Show that $\pi_n(X_1 \times X_2) \cong \pi_n(X_1) \times \pi_n(X_2)$ for every $n \geq 0$.*

(b) *Compute the fundamental group of the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$, and give an explicit generating set.*

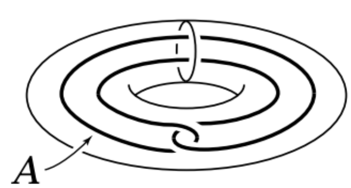
Solution. □

Problem 4 (Hatcher 1.1.7)**.** *Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, r) = (\theta + 2\pi r, r)$. Observe that f is the identity on the boundary $S^1 \times \{0, 1\}$. Show that there is a homotopy f_t from f to the identity such that f_t is the identity on $S^1 \times \{0\}$ for all t, but no homotopy where f_t is the identity on $S^1 \times \{0, 1\}$ for all t. Hint: consider the image of an arc $\{\theta_0\} \times I$ under f.* [2]

Solution. □

Problem 5 (Hatcher 1.1.16)**.** *Show there is no retract $r : X \to A$ in the following cases. Hint: this should be a simple matter of algebra.*

(a) *$X = S^1 \times D^2$ and A the circle shown in the figure.*



(b) *X the Möbius band and A its boundary circle.*

[1] A group with a topology so that the multiplication and inversion maps are continuous.

[2] The homeomorphism f is important in studying the *mapping class group* of a surface S, which is the group of homotopy classes of homeomorphisms $S \to S$.

Solution. □

Problem 6 (Hatcher 1.1.19)**.** *Let $X = S^1$ with basepoint x_0. Show that every based loop in X is homotopic to a finite sequence of arcs traversed monotonically. Use a further homotopy to deduce that $\pi_1(S^1)$ is cyclic. (Of course in class we proved that $\pi_1(S^1) \cong \mathbb{Z}$. Your argument here will not rule out the possibility that $\pi_1(S^1)$ is finite.)*

Solution. □

Problem 7 (Hatcher 1.1.20)**.** *Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are the identity map.*

(a) Show that for any $x_0 \in X$ the loop $\alpha(t) := f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$.

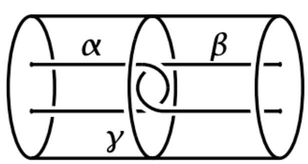
(b) Let $\mathrm{Homeo}(X)$ denote the group of homeomorphisms of X. Consider the evaluation map $\epsilon : \mathrm{Homeo}(X) \to X$ defined by $\epsilon(f) = f(x_0)$. Conclude that $\epsilon_* : \pi_1(\mathrm{Homeo}(X), \mathrm{id}) \to \pi_1(X, x_0)$ has image contained in the center of $\pi_1(X, x_0)$.

Solution. □

Problem 8. *Let $S_g = T^2 \# \cdots \# T^2$ (g times), and let $H_g = \mathbb{R}P^2 \# S_g$. Prove that no two of these surfaces are homeomorphic. Hint: compute the fundamental group and its abelianization (you will need to explain why the abelianization is what you claim it is).*

Solution. □

Problem 9 (Hatcher 1.2.10)**.** *Consider two arcs α, β embedded in $I \times D^2$ as in the figure below. The loop γ is obviously nullhomotopic in $I \times D^2$. Show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$. Hint: Untangle α and β and identify $I \times D^2 \setminus (\alpha \cup \beta)$ with $I \times (D^2 \setminus \{2 \text{ points}\})$; keep track of γ along the way.*

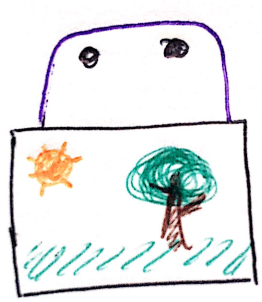


Solution. □

Problem 10 (Hatcher 1.2.11)**.** *The mapping torus M_f of a map $f : X \to X$ is the quotient of $X \times I$ by identifying $(x, 0)$ and $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(M_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. Hint: one way to do this is to regard M_f as built from $X \vee S^1$ by attaching cells.*

Solution. □

Problem 11. *Consider a picture frame hanging on a wall, supported by two nails and a string as shown below.*



(a) *Observe that the picture will remain hanging even if either one (but not both) of the nails is removed. Find a way to wind the string around the nails so that the picture will fall if either nail is removed (the string must be attached to the frame in the same way).*

(b) *Suppose now there are three nails. Find a way to hang the string so that it falls if any two nails are pulled from the wall, but not before.*

(c) *For three nails, find a way to hang the string so that it falls if any nail is pulled from the wall. Express your answer using algebra instead of a drawing.*

Solution. □

Problem 12 (Bonus). *The point of this problem is to learn, for a knot $K \subset \mathbb{R}^3$ a way to compute a presentation for $\pi_1(\mathbb{R}^3 \setminus K)$. The presentation discussed here called Wirtinger's presentation.*

(a) *Read Hatcher's explanation of the Wirtinger presentation (Exercise* 22 *in Section* 1.2*), or read it somewhere else, e.g. there is a nice explanation in Stillwell's* Classical Topology and Combinatorial Group Theory*, Section* 4.2.3.

(b) *Give the Wirtinger presentation for the standard projection of the trefoil knot. Relate $\pi_1(\mathbb{R}^3 \setminus K)$ to a braid group. Give a presentation for $\pi_1(\mathbb{R}^3 \setminus K)$ when K is the figure-8 knot.*

(c) *Use the Wirtinger presentation to show that the abelianization of $\pi_1(\mathbb{R}^3 \setminus K)$ is $\mathbb{Z}$ for any knot K.*

Solution. □

Problem 13 (Bonus). *Let $p = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial with coefficients $a_i \in \mathbb{C}$. For each $r > 0$, consider the map $C_r : [0,1] \to \mathbb{C} \setminus \{0\}$ defined by $t \mapsto re^{2\pi i t}$ (parameterizing the circle of radius r).*

(a) *Suppose that $p(0) \neq 0$. Show that if r is sufficiently small, then $p \circ C_r$ is homotopic to the constant map $t \mapsto p(0)$ through maps contained in $\mathbb{C} \setminus \{0\}$. (The crucial thing is to ensure that the homotopy never passes through $0 \in \mathbb{C}$.)*

(b) *Show that if r is sufficiently large, then $p \circ C_r$ is homotopy equivalent to $C_r^n : t \mapsto (re^{2\pi i t})^n$ through maps contained in $\mathbb{C} \setminus \{0\}$.* [3]

[3] Hint: use a straight-line homotopy; the key is to ensure that the homotopy never passes through $0 \in \mathbb{C}$; you will need to do a little analysis.

(c) Explain how (a) and (b) can be used to prove that p has a root.

Solution. □