Homework 1

Math 241

Due September 16, 2019 by 5pm

Topics covered: Cell complexes, homotopy, deformation retracts, quotient by contractible theorem, homotopic attachments theorem, homotopy extension property

Instructions:

- This assignment must be submitted on Canvas by the due date.
- You are encouraged to collaborate with other students, but you must write your own solutions. If you do collaborate, please mention this near the corresponding problems.
- Most problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.

Problem 1 (Bredon I.14.3)**.** *Let X be the union of the torus with two disks, one spanning a longitudinal circle, and the other spanning a meridian circle. Show that X is homotopy equivalent to S^2.*

Solution. □

Problem 2 (Hatcher 0.3)**.** (a) *Show the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of homotopy equivalences is a homotopy equivalence. Deduce that homotopy equivalence of spaces $X \simeq Y$ is an equivalence relation.*

(b) *Show that the relation $f \simeq g$ if $f, g : X \to Y$ are homotopic is an equivalence relation.*

(c) *Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.*

Solution. □

Problem 3 (Hatcher 0.16)**.** *Show that $S^\infty = \bigcup S^n$ is contractible. Note that we can express $x \in S^\infty$ as $(x_1, x_2, \ldots) \in \mathbb{R}^\infty$, where $\sum x_i^2 = 1$ and all but finitely many coefficients are 0. Hint: first homotope S^∞ to its equator $\{x_1 = 0\}$; then homotope the equator to the north pole $(1, 0, 0, \ldots)$.*

Solution. □

Problem 4 (Hatcher 0.23)**.** *Let X be a cell complex that is the union of two contractible subcomplex whose intersection is contractible. Show that X is contractible.*

Solution. □

Problem 5 (Hatcher 0.21)**.** *Let X be a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point. Show that X is homotopy equivalent to a wedge sum of S^1's and S^2's. Hint: find a tree to collapse to a point.*

Solution. □

Problem 6. *Consider $(X, A) = ([0, 1], \{1/n\} \cup \{0\})$. Let $g : A \to CA$ be the constant map at the cone point, and let $h : A \to CA$ be the inclusion. Show $g \simeq h$. Show g extends to X but h does not. Conclude (X, A) does not have HEP.*

Solution. □

Problem 7. *Let $A \subset X$ be a closed subspace.*

(a) *Show that the following statements are equivalent. (i) For every Y, every map $g : A \to Y$ extends to a map $G : X \to Y$. (ii) There exists a retract $X \to A$.*

(b) *Show that the following statements are equivalent. (i) For every Y, every map $A \to Y$ extends to X, and if $F, G : X \to Y$ are maps whose restriction to A are homotopy equivalent, then $F \simeq G$. (ii) A is a weak deformation retract of X, i.e. there is a homotopy from 1_X to a retract $X \to A$.*

Solution. □

Problem 8 (Hatcher 0.18). *The join $X * Y$ of two spaces is the quotient of $X \times Y \times I$ by the equivalence relation $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x, y', 1)$. I.e. we collapse $X \times Y \times \{0\}$ and $X \times Y \times \{1\}$ to X and Y, respectively. Show $S^1 * S^1 \simeq S^3$ and more generally $S^n * S^m \simeq S^{n+m+1}$. Hint: observe that $\partial(D^a \times D^b) = \partial D^a \times D^b \cup D^a \times \partial D^b$.*

Solution. □

Problem 9 (Hatcher 0.14). *Given $e_0, e_1, e_2 \geq 1$ such that $e_0 - e_1 + e_2 = 2$, construct a cell structure of S^2 with e_i cells of dimension i.*

Solution. □

Problem 10 (Hatcher 0.15). *Enumerate the sub-complexes of S^∞ with the standard cell structure that has S^n as its n-skeleton (there are two cells of each dimension).*

Solution. □

Problem 11 (Hatcher 0.6).

(a) *Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0, 1] \times \{0\}$ together with the vertical segments $r \times [0, 1 - r]$ for $r \in \mathbb{Q} \cap [0, 1]$. Show that X deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point. Hint: First prove that if X deformation retracts to $x \in X$, then for each neighborhood U of x, there exists a neighborhood $V \subset U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic.*

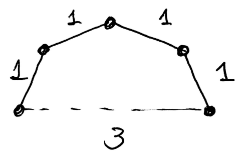
(b) *Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavy line in the figure. Show that there is a weak deformation retract of Y onto Z. Deduce that Y is contractible, although it does not retract to any point. Hint: move each point at unit speed "to the right."*

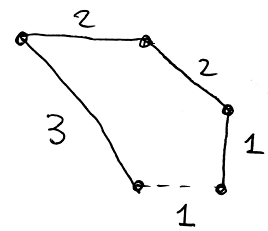
Solution. □

Problem 12 (Bonus). *In this problem you will determine the configuration spaces of two different linkages, each with four rods, attached in a line, with fixed endpoints.*

(a) *Consider the linkage L where each of the rods has unit length and that the distance between the two endpoints is 3. Show that $\mathcal{C}(L)$ is homeomorphic to S^2.*

(b) *Consider the linkage L where the rods have length $3, 2, 2, 1$, and the two endpoints have distance 1. Show that $\mathcal{C}(L)$ is homeomorphic to a surface of genus 2.*





Solution. □

Problem 13 (Bonus)**.** *Prove that* $\mathrm{GL}_n(\mathbb{R})$ *deformation retracts to* $O(n)$*. Hint: you may want to several tools from linear algebra: polar decomposition, the Gram-Schmidt procedure, and the identification of the set of positive matrices with the set of inner products.*

Solution. □