Homework 5

Math 25b

Due March, 7 2018

Topics covered: Derivatives, inverse function theorem, implicit function theorem

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds.* I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Ellen

Problem 1 (Munkres, 7-3). *Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by the equation*

$$F(x, y) = f(x, y, g(x, y)).$$

(a) Find DF in terms of the partials of f and g.

(b) If $F(x, y) = 0$ for all (x, y), find $D_1 g$ and $D_2 g$ in terms of the partials of f.

Solution. □

Problem 2 (Munkres, 8-1). *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation*

$$f(x, y) = (x^2 - y^2, 2xy).$$

(a) Show that f is injective on $A = \{(x, y) : x > 0\}$. Hint. If $f(x, y) = f(a, b)$, then $|f(x, y)| = |f(a, b)|$.

(b) What is the set $B = f(A)$.

(c) If g is the inverse function, find $Dg(0, 1)$.[1]

Solution. □

[1] After you do this problem, it's worth thinking about how much simpler life was last semester when we were dealing with *linear* functions!

2 For Charlie

Problem 3 (Munkres, 8-3)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be given by the equation* $f(x) = |x|^2 \cdot x$*. Show that* f *has derivative of all orders and that* f *carries the unit ball* $B_1 = \{x \in \mathbb{R}^n : |x| \leq 1\}$ *to itself in an injective fashion. Show however that the inverse function is not differentiable at 0.*

Solution. □

Problem 4 (Munkres, 8-4)**.** *This exercise is a reality check of your concrete understanding of inverse function theorem (and chain rule). Define* $g : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *by*

$$g(x, y) = (2ye^{2x}, xe^y) \quad \textit{and} \quad f(x, y) = (3x - y^2, 2x + y, xy + y^3).$$

(a) Show that there is a neighborhood of $(0, 1)$ *that* g *carries in a bijective fashion to a neighborhood of* $(2, 0)$*.*

(b) Find $D(f \circ g^{-1})$ *at* $(2, 0)$*.*

Solution. □

3 For Michele

Problem 5 (Munkres, 8-5). *Let A be open in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}^n$ be C^1. Assume $Df(x)$ is invertible for $x \in A$. Show that $B = f(A)$ is open, even if f is not injective. Hint: You have to find the place in the proof from class where we used injectivity in a nontrivial way!*

Solution. □

Problem 6 (Spivak, CoM 2-37).

(a) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is not 1-1. Hint: If, for example, $D_1 f(x, y) \neq 0$ for all (x, y) in some open set A, consider $g : A \to \mathbb{R}^2$ defined by $g(x, y) = (f(x, y), y)$.*

(b) *Generalize this result to the case of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$.* [2]

Solution. □

Problem 7 (Spivak, CoM 2-38). *Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that* $\det f'(x, y) \neq 0$ *for all (x, y) but f is not 1-1.*

Solution. □

[2] This problem is related to the Implicit Function Theorem. It can be proved using this theorem, but I'd like you to *not* quote it. It might be helpful to read the proof of the implicit function theorem in Munkres. Additional information/intuition available at office hours.

4 For Natalia

Problem 8 (2-39)**.** *Use the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

to show that continuity of the derivative cannot be eliminated from the hypothesis of Theorem 2-11 (the inverse function theorem). Hint. Show that f *is differentiable, but* f' *is not continuous, and that* f *is not injective in any neighborhood of 0. The last step is the tricky part – it might help to remember what* f', f'' *tell us about* f*.*

Solution. □

Problem 9 (Spivak, CoM 2-40)**.** *Use the implicit function theorem to re-do Problem 2-15(c). (It should be less painful this time!)* [3]

Solution. □

[3] If you set up the implicit function theorem in the right way, this will be extremely painless.