Homework 4

Math 25b

Due February, 21 2018

Topics covered: Mean value theorem, derivatives in higher dimensions

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds.* I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Michele

Problem 1 (Spivak, CoM 2-1). *Give an ϵ-δ proof that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it is continuous at a. Hint: use problem 1-10.* [1]

Solution. □

Problem 2 (Spivak, C 11-26). *Suppose $f'(x) \geq M$ for all $x \in [0, 1]$. Show that there is an interval of length $\frac{1}{4}$ on which $|f| \geq M/4$. Hint: use MVT.*

Solution. □

Problem 3 (Spivak, C 11-57). *In this problem you prove that (usually) $(x + y)^n \neq x^n + y^n$. The faulty assertion that $(x + y)^n = x^n + y^n$ is sometimes called the "freshman dream".*[2]

(a) *Assume $y \neq 0$ and n is even. Prove that $x^n + y^n = (x + y)^n$ only when $x = 0$. Hint: Suppose the statement holds for some $x_0 \neq 0$ and use Rolle's theorem.*

(b) *Prove that if $y \neq 0$ and n is odd, then $x^n + y^n = (x + y)^n$ only if $x = 0$ or $x = -y$. Hint: What does Rolle's say in this case? Why is this good enough?*

Solution. □

[1] Munkres Ch. 2 Thm. 5.2 gives a proof using algebra of limits, but I want you to give an ϵ-δ proof. It's good practice.

[2] Presumably referring to freshmen at a less reputable institution, e.g. Yale.

2 For Charlie

Problem 4 (Spivak, CoM 2-29). *Recall from class that for $f : \mathbb{R}^n \to \mathbb{R}$, we define the directional derivative of f at a in the direction v by*

$$D_v f(a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t},$$

if the limit exists.

(a) *Show that $D_{cv} f(a) = c D_v f(a)$ for $c \in \mathbb{R}$.*

(b) *If f is differentiable at a, show that $D_v f(a) = Df(a)(v)$ and therefore $D_{u+v} f(a) = D_u f(a) + D_v f(a)$. Hint: part (a) might be helpful.*

Solution. □

Problem 5 (Spivak, CoM 2-4, 2-5, 2-30). *Let g be a continuous real-valued function on the unit circle $S^1 = \{z \in \mathbb{R}^2 : |z| = 1\}$ such that $g(0,1) = g(1,0) = 0$ and $g(-z) = -g(z)$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(z) = \begin{cases} |z| \cdot g\left(\frac{z}{|z|}\right) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

(a) *Assume $z \in \mathbb{R}^2$ and $|z| = 1$. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = f(tz)$, show that h is differentiable.*

(b) *Show that $D_v f(0)$ exists for all v (and find it explicitly).*

(c) *Show that f is not differentiable at 0 unless $g = 0$.*

(d) *Observe that $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

is a function of the kind considered in (a), so that f is not differentiable at (0,0).

Solution. □

Problem 6 (Spivak, CoM 2-7). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0. Hint: First guess the derivative.*

Solution. □

3 For Ellen

Problem 7 (Spivak, CoM 2-13). *Define $IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $IP(x, y) = \langle x, y\rangle$ (the standard inner product).*

(a) Find $D(IP)(a, b)$.

(b) If $f, g : \mathbb{R} \to \mathbb{R}^n$ are differentiable and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \langle f(t), g(t)\rangle$, show that

$$h'(a) = \langle g(a), f'(a)\rangle + \langle f(a), g'(a)\rangle.$$

(c) If $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable and $|f(t)| = 1$ for all t, show that $\langle f(t), f'(t)\rangle = 0$.[3]

Solution. □

Problem 8 (Spivak, CoM 2-14). *Let E_i, $i = 1, \dots, k$ be Euclidean spaces of various dimensions. A function $f : E_1 \times \cdots \times E_k \to \mathbb{R}^p$ is called multilinear if it is linear in each coordinate, i.e. if for each i and choice of $x_j \in E_j$, $j \neq i$, the function $g : E_i \to \mathbb{R}^p$ defined by $g(x) = f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k)$ is linear.*

(a) If f is multilinear and $i \neq j$, show that for $h = (h_1, \dots, h_k)$, with $h_\ell \in E_\ell$, we have

$$\lim_{h\to 0} \frac{|f(a_1, \dots, h_i, \dots, h_j, \dots, a_k)|}{|h|} = 0$$

Hint: If $g(x, y) = f(a_1, \dots, x, \dots, y, \dots, a_k)$, then g is bilinear, so you can reduce to showing that for a bilinear map $g : E \times F \to \mathbb{R}$ one has $\lim_{h\to 0} \frac{|g(h_1,h_2)|}{|h|} = 0$.

(b) Prove that

$$Df(a_1, \dots, a_k)(x_1, \dots, x_k) = \sum_{i=1}^{k} f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_k).$$

Solution. □

[3] Later we'll use this exercise to compute the *tangent space* of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

4 For Natalia

Problem 9 (Spivak, CoM 2-15). *Regard an $n \times n$ matrix as a point in the n-fold product $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ by considering each column as a vector in $\mathbb{R}^n$.*

(a) *Prove that* $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ *is differentiable and*

$$D(\det)(a_1, \dots, a_n)(x_1, \dots, x_n) = \sum_{i=1}^{n} \det\left(a_1 | \cdots | x_i | \cdots | a_n\right).$$

(b) *If $a_{ij} : \mathbb{R} \to \mathbb{R}$ are differentiable and $f(t) = \det(a_{ij}(t))$, show that*

$$f'(t) = \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(t) & \cdots & a'_{1j}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n1}(t) & \cdots & a'_{nj}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

(c) *If $\det(a_{ij}(t)) \neq 0$ for all t and $b_1, \dots, b_n : \mathbb{R} \to \mathbb{R}$ are differentiable, let $s_1, \dots, s_n : \mathbb{R} \to \mathbb{R}$ be the functions such that $s_1(t), \dots, s_n(t)$ are the solutions of the equations*

$$\sum_{j=1}^{n} a_{ij}(t) s_j(t) = b_i(t) \quad i = 1, \dots, n.$$

Show that s_j is differentiable and find $s'_j(t)$.

Solution. □

Problem 10 (Spivak, CoM 2-10). *Find $Df(x, y)$ for the following:*

(a) $f(x, y) = \sin(x \sin y)$

(b) $f(x, y) = \sin(xy)$

(c) $f(x, y) = (\sin(xy), \sin(x \sin y))$

Solution. □

Problem 11 (Spivak, CoM 2-24). *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

(a) *Show that $D_2 f(x, 0) = x$ for all x and $D_1 f(0, y) = -y$ for all y. Hint: You don't actually have to compute too much to solve this part.*

(b) *Show that $D_{1,2} f(0, 0) \neq D_{2,1} f(0, 0)$.*

Solution. □