Homework 10

Math 25b

Due April 25, 2018

Topics covered: differential forms, chains, Stokes' theorem

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Most problems from this assignment come from Spivak's *Calculus* or Spivak's *Calculus on manifolds* or Munkres' *Analysis on manifolds*. I've indicated this next to the problems (e.g. Spivak, CoM 1-2 means problem 2 of chapter 1 from Calculus on Manifolds).
- Any result that we proved in class can be freely used on the homework. If there's a result that we haven't stated in class that you want to use, then you have to prove it. If there's a result that we stated in class, but haven't proven, it's best to ask for clarification.

1 For Ellen

Notation. In the problems 1-4, we'll use the following notation.

- $A = \mathbb{R}^2 \setminus \{0\}$ and $B = A \setminus X_+$, where X_+ is the positive x-axis.
- $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ and $\phi : B \to \mathbb{R}$ is the function from HW9#9 such that $\omega = d\phi$ on B.
- $c_{R,n} : [0,1] \to A$ is the 1-cube defined by $c_{R,n}(t) = (R\,\cos 2\pi nt, R \sin 2\pi nt)$.

Problem 1 (Spivak, CoM 4-26)**.** *Compute* $\int_{c_{R,n}} \omega = 2\pi n$ *and use Stokes' theorem to conclude that* $c_{R,n} \neq \partial c$ *for any* 2*-chain* c *on* A.

Solution. □

Problem 2 (Spivak, CoM 4-24)**.** *Let* c *is a singular* 1*-cube in* A *with* $c(0) = c(1)$*. In this problem, you show that there is an integer* n *such that* $c - c_{1,n} = \partial b$ *for some* 2*-cube* b*.* [1]

(a) *Show that* $\int_c \omega = 2\pi n$ *for some integer* n*. For simplicity, assume that* $c(0) = c(1)$ *is on the* x*-axis and that there are numbers* $0 = t_1 < t_2 < \cdots < t_m = 1$ *so that* $c(t_i) \in X_+$ *for each* i*, and* $c(t) \in B$ *for* $t \notin \{t_1, \ldots, t_m\}$*. Hint: use that* $\omega = d\phi$ *on* B *and use FTC.*

(b) *Take* n *as in (a). Find a* 2*-cube* $b : [0,1]^2 \to A$ *so that* $\partial b = c - c_{1,n}$*. Hint: one approach is to define functions* $r(t) = |c(t)|$ *and* $\theta(t) = \int_0^t c^*\omega$ *and to do a "straight-line interpolation" between* $r(t)$ *and 1 and between* $\theta(t)$ *and* $2\pi nt$*.*

Solution. □

Problem 3 (Spivak, CoM 4-27)**.** *Show that the integer* n *of problem* 4-24 *is unique. This integer is called the* winding number *of* c *around* 0*. Hint: use the previous two problems.*

Solution. □

[1] Call a 1-cube whose endpoints are equal a *loop.* In essence this problem says that "up to deformation" (as encoded by the boundary map), there is a single invariant of a loop in A: the *winding number* (defined in Problem 3). This is a basic and important result about the topology of A.

2 For Natalia

Problem 4 (Spivak, CoM 4-30)**.** [2]*Suppose η is any 1-form on A such that $d\eta = 0$. Prove*

$$\eta = \lambda\,\omega + dg$$

for some $\lambda \in \mathbb{R}$ and $g : \mathbb{R}^2 \setminus 0 \to \mathbb{R}$. Hint: First find λ. Use 4-29 (problem 5) to write $c_{R,1}^*(\eta) = 2\pi\lambda_R\,dt + d(g_R)$ for some λ_R and g_R with $g_R(0) = g_R(1)$, and show that all numbers λ_R have the same value λ (using HW 9#10). Then use 4-32 (problem 8) to show that $\eta - \lambda\omega$ is exact (see the footnote to that problem). For this last part you'll probably also want to use 4-24 (problem 2). [3]

Solution. □

Notation. In the following problems ω is back to denoting an arbitrary form.

Problem 5 (Spivak, CoM 4-29)**.** *Let $\omega = f(x)\,dx$ be a 1-form on $[0, 1]$.*

(a) Is ω necessarily closed? exact? Explain your answer.

(b) Assume $f(0) = f(1)$. Show that there is a unique number λ such that $\omega - \lambda\,dx = dg$ for some function g with $g(0) = g(1)$. Hint: integrate $\omega - \lambda\,dx = dg$ on $[0, 1]$ to find λ.

Solution. □

Problem 6 (Spivak, CoM 4-31)**.** *Fix an open set $U \subset \mathbb{R}^n$ and fix a k-form $\omega \in \Omega^k(U)$.*

(a) Show that if $\omega \neq 0$ then there is a chain c such that $\int_c \omega \neq 0$.

(b) Use (a), Stokes' theorem, and $\partial^2 = 0$ to prove $d^2\omega = 0$.

Solution. □

[2] Before approaching this problem, you should first solve Problems 5 and 8.

[3] As a consequence of this problem, you've computed the 1-*dimensional de Rham cohomology vector space of* A, often denoted $H^1_{dR}(A)$. This vector space is defined as the quotient of the vector space of closed 1-forms by the subspace of exact 1-forms. The fact that $H^1_{dR}(A) \neq 0$ is indicative of the fact that A has nontrivial topology. For $k \geq 0$ and $U \subset \mathbb{R}^n$ open, the De Rham cohomology $H^k_{dR}(U)$ (defined similarly as closed k-forms modulo exact k-forms) gives a precise measure of the extent to which the Fundamental Theorem of Calculus fails to extend to k-forms on U.

3 For Michele

Problem 7 (Spivak, CoM 2-21 and 3-34). *Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be C^1. Define $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x,y) = \int_0^x g_1(t,0)\,dt + \int_0^y g_2(x,t)dt.$$

(a) Show that $D_2 f(x, y) = g_2(x, y)$.

(b) Assume now that $D_1 g_2 = D_2 g_1$, and show that $D_1 f(x, y) = g_1(x, y)$. Hint: You may use Munkres Theorem 39.1, which concerns differentiating "under the integral".

Solution. □

Problem 8 (Spivak, CoM 4-32). *Let c_1, c_2 be 1-cubes in $\mathbb{R}^2$ with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$ (in other words, viewing c_1 and c_2 as curves in the plane, they have the same endpoints).*

(a) Show that there is a 2-cube c such that $\partial c = c_1 - c_2 + c_3 - c_4$, where c_3 and c_4 are degenerate, meaning $c_3([0, 1])$ and $c_4([0, 1])$ are points.

(b) Conclude that $\int_{c_1} \omega = \int_{c_2} \omega$ if ω is exact. Give a counterexample on $\mathbb{R}^2 \setminus 0$ if ω is merely closed.

(c) Prove the converse of (b): If ω is a 1-form on a subset of $\mathbb{R}^2$ and $\int_{c_1} \omega = \int_{c_2} \omega$ for all c_1 and c_2 with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$, then ω is exact. Hint: use problem 7. [4]

Solution. □

Problem 9. *True or False. Explain your answer.*

(a) If $\alpha \in \Lambda^1(\mathbb{R}^3)$, then $\alpha \wedge \alpha = 0$.

(b) If $\beta \in \Lambda^2(\mathbb{R}^4)$, then $\beta \wedge \beta = 0$.

(c) There exists $\omega \in \Omega^1(\mathbb{R}^3)$ so that $\omega \wedge d\omega = 0$.

(d) There exists $\omega \in \Omega^1(\mathbb{R}^3)$ so that $\omega \wedge d\omega \neq 0$.

Solution. □

[4] Remark: The condition $\int_{c_1} \omega = \int_{c_2} \omega$ for every pair c_1, c_2 with the same endpoints is equivalent to saying that $\int_c \omega = 0$ for every 1-cube with $c(0) = c(1)$. This alternate formulation is useful for problem 4.

4 For Charlie

Problem 10. *Fix* $0 < a < b$ *and define* $c(r,\theta) = (r\cos\theta, r\sin\theta)$ *for* $a \le r \le b$ *and* $0 \le \theta \le 2\pi$. *(What is the image of* c*?) Put* $\omega = x^3\,dy$ *and compute both*

$$\int_c d\omega \quad \textit{and} \quad \int_{\partial c} \omega$$

to verify that they are equal.

Solution. □

Problem 11 (Munkres, 31-2,4)**.** *Munkres Theorem 31.1 has the following commutative diagram for an open set* $A \subset \mathbb{R}^n$.

$$\begin{array}{ccc} \textit{Scalar fields on } A & \xrightarrow{\alpha_0} & \Omega^0(A) \\ \Big\downarrow \textit{grad} & & \Big\downarrow d \\ \textit{Vector fields on } A & \xrightarrow{\alpha_1} & \Omega^1(A) \end{array}$$

$$\begin{array}{ccc} \textit{Vector fields on } A & \xrightarrow{\beta_{n-1}} & \Omega^{n-1}(A) \\ \Big\downarrow \textit{div} & & \Big\downarrow d \\ \textit{Scalar fields on } A & \xrightarrow{\beta_n} & \Omega^n(A) \end{array}$$

[5]*Saying this diagram "commutes" means that* $d \circ \alpha_0 = \alpha_1 \circ \textit{grad}$ *and* $d \circ \beta_0 = \beta_1 \circ \textit{div}$ *(i.e. following the arrows around from top-left to bottom-right in either direction give the same function). We discussed maps like this in the special case* $n = 3$.

(a) Note that in the case $n = 2$, *Theorem 31.1 gives two maps* α_1, β_1 *from vector fields to 1-forms. Compare them.*

For $\mathbb{R}^4$, *there is a way of translating theorems about forms into more familiar language if one allows oneself to use "matrix fields" as well as vector fields and scalar fields. You'll explore this here. The complications may help understand why the language of forms was invented to deal with* $\mathbb{R}^n$ *in general.*

A matrix $B \in M_n(\mathbb{R})$ *is called skew-symmetric if* $B^t = -B$. *Let* $S(\mathbb{R}^4)$ *be the set of smooth functions* $H : \mathbb{R}^4 \to M_4(\mathbb{R})$ *so that* $H(x)$ *is skew-symmetric for each* $x \in \mathbb{R}^4$. *Denoting the* (i,j) *entry of* $H(x)$ *by* $h_{ij}(x)$, *define* $\gamma_2 : S(\mathbb{R}^4) \to \Omega^2(A)$ *by*

$$\gamma_2(H) = \sum_{i<j} h_{ij}\,dx_i \wedge dx_j.$$

(b) Show that γ_2 *is a linear isomorphism.*

[5]A scalar field is another term for a function $f : A \to \mathbb{R}$.

(c) *Let $\alpha_0, \alpha_1, \beta_3, \beta_4$ be defined as in Theorem 31.1 (see previous problem). Your job in this part is to define operators "twist" and "spin" as in the diagram*

$$\begin{array}{ccc}
\textit{Vector fields on } \mathbb{R}^4 & \xrightarrow{\alpha_1} & \Omega^1(\mathbb{R}^4) \\
\Big\downarrow \textit{twist} & & \Big\downarrow d \\
S(\mathbb{R}^4) & \xrightarrow{\gamma_2} & \Omega^2(\mathbb{R}^4) \\
\Big\downarrow \textit{spin} & & \Big\downarrow d \\
\textit{Vector fields on } \mathbb{R}^4 & \xrightarrow{\beta_3} & \Omega^3(\mathbb{R}^4)
\end{array}$$

so that $d \circ \alpha_1 = \gamma_2 \circ$ twist and $d \circ \gamma_2 = \beta_3 \circ$ spin. (These operators are facetious analogues in $\mathbb{R}^4$ of the operator "curl" in $\mathbb{R}^3$.)

Solution. □

⋆ **You've made it to the end. Woohoo!** ⋆