Homework 8

Math 25a

Due November 16, 2018

Topics covered (lectures 15-17): eigenvectors, polynomials, satisfied polynomials

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Beckham

Problem 1 (Axler 5.A.18,20)**.** *Find all eigenvalues and eigenvectors of the following linear operators in* $L(F^\infty)$*.*

(a) $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$

(b) $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$

Solution. □

Problem 2 (Axler 5.B.3)**.** *Suppose* $T \in L(V)$ *and* $T^2 = I$ *and* -1 *is not an eigenvalue of* T*. Prove that* $T = I$*. Hint: satisfied polynomials.*

Solution. □

Problem 3 (Axler 5.A.11-12)**.** *Let* $D : \mathrm{Poly}(\mathbb{R}) \to \mathbb{R}$ *be the derivative.*

(a) Find all eigenvalues and eigenvectors of D*.*

(b) Consider $T : \mathrm{Poly}_4(\mathbb{R}) \to \mathrm{Poly}_4(\mathbb{R})$ *defined by* $T(p) = x \cdot D(p)$*. Find all eigenvalues and eigenvectors of* T*.*

Solution. □

2 For Davis

Problem 4 (Treil 4.1.11)**.** *Fix $A \in M_n(\mathbb{C})$. Recall that the trace of $A = (a_{ij})$ is*

$$\operatorname{tr}(A) = a_{11} + \cdots + a_{nn}.$$

Show that $\operatorname{tr}(A)$ is the sum of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A as follows.

(a) *Compute the coefficient of t^{n-1} in the right side of the equality*

$$\det(A - tI) = (\lambda_1 - t) \cdots (\lambda_n - t).$$

(b) *Show that $\det(A - tI)$ can be represented as*

$$\det(A - tI) = (a_{11} - t) \cdots (a_{nn} - t) + q(t)$$

where $q(t)$ is a polynomial of degree at most $n - 2$.

(c) *Conclude $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$ by comparing coefficients on t^{n-1}.*

(d) *Consider the matrix*

$$A = \begin{pmatrix} 51 & -12 & 21 \\ 60 & -40 & -28 \\ 57 & -68 & 1 \end{pmatrix}.$$

Two of the eigenvalues of A are -48 and 24. Without using a computer or writing anything down, find the third eigenvalue.

Solution. □

Problem 5 (Axler 5.A.15)**.** *Fix $S, T \in L(V)$ and assume S is invertible.*

(a) *Prove that T and STS^{-1} have the same eigenvalues.*

(b) *How are the eigenvectors of T and the eigenvectors of STS^{-1} related?*

Solution. □

Problem 6 (Axler 5.A.24)**.** *Let $A \in M_n(F)$. Let $T \in L(F^n)$ be the linear operator given by $Tx = Ax$.*

(a) *Suppose the sum of the entries in each row of A equals k. Prove that k is an eigenvalue of T.*

(b) *Suppose the sum of the entries in each column of A equals k. Prove that k is an eigenvalue of T.*

Solution. □

3 For Joey

Problem 7 (Treil 4.1.7-9). *(a) Show that the characteristic polynomial of a block triangular matrix* $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$*, where* A, B *are square matrices, is* $\det(A - xI)\det(B - xI)$*. Hint: use a problem from HW7.*

(b) Let $v_1, \ldots, v_n$ *be a basis for* V*. Assume that* $v_1, \ldots, v_k$ *are eigenvectors for* T *with eigenvalue* λ*, i.e.* $Tv_j = \lambda v_j$ *for* $j = 1, \ldots, k$*. Show that in this basis the matrix of* T *has block triangular form*

$$\begin{pmatrix} \lambda I_k & * \\ 0 & B \end{pmatrix}$$

where I_k *is the* $k \times k$ *identity matrix and* $B \in M_{n-k}(F)$*.*

(c) Use (a) and (b) to prove that the geometric multiplicity is at most the algebraic multiplicity.

Solution. □

Problem 8 (Axler 5.A.26). *Suppose that* $T \in L(V)$ *is such that every nonzero vector in* V *is an eigenvector of* T*. Prove that* $T = cI$ *is a scalar multiple of the identity. Hint: it might help to first prove that if* u, v *are eigenvectors of* T *such that* $u + v$ *is also an eigenvector of* T*, then* u *and* v *have the same eigenvalue.*

Solution. □

Problem 9 (Axler 5.A.28). *Fix finite dimensional* V *and assume* $\dim V \geq 3$*. Suppose that* $T \in L(V)$ *and that every 2-dimensional subspace* $U \subset V$ *is invariant*[1] *under* T*. Show that* $T = cI$ *for some* $c \in F$*. Hint: start with* $v \in V$ *and show directly that* $Tv = \lambda v$ *for some* λ*.*

Solution. □

[1] A subspace $U \subset V$ is called <u>invariant</u> under T if $T(u) \in U$ for all $u \in U$. For example, if U is 1-dimensional, then this is equivalent to the nonzero vectors in U being eigenvectors. If $U = \text{span}(u, w)$ is 2-dimensional, then U is invariant means that $T(u) = au + bw$ and also $T(w) = cu + dw$ for some $a, b, c, d \in F$.

4 For Laura

Problem 10. *Suppose $T \in L(\mathbb{R}^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ so that $Tx - 9x = (-4, 5, \sqrt{7})$.*

Solution. □

Problem 11 (Axler 5.A.23)**.** *Suppose V is finite dimensional and $S, T \in L(V)$. Prove that ST and TS have the same eigenvalues. (Hint: You will need to use the assumption that V is finite dimensional!)*

Solution. □

Problem 12. *Recall the Cayley–Hamilton theorem: $A \in M_n(F)$ satisfies its characteristic polynomial $p_A = \det(A - xI)$. Prove this in the case when A is diagonalizable.*

Solution. □