Homework 6

Math 25a

Due October 26, 2018

Topics covered (lectures 11-12): linear systems, row operations, elementary matrices, invertibility/inverses

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Joey

Problem 1. *A matrix* $A \in M_n(F)$ *is invertible if there exists a matrix* B *so that* $AB = BA = I$*. Prove that if* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$ *and* $ad - bc \neq 0$*, then* A *is invertible. Find the inverse of* A*. (Option 1: apply row reduction to* $\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$*. Option 2: write* $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ *and set up a system of equations to solve for the coefficients of* B*.) Hint: it may help to consider the cases* $a \neq 0$ *and* $a = 0$ *separately.*

Solution. □

Problem 2. *Let* $F = \mathbb{Z}/2\mathbb{Z}$*.*

(a) *Write down all the invertible matrices in* $M_2(F)$*.*

(b) *Find an invertible matrix* $A \in M_2(F)$ *so that* $A^2 \neq I$*, but* $A^3 = I$*.*

Solution. □

Problem 3 (Axler 3.D.9)**.** *Assume* V *is finite dimensional and let* $S, T \in L(V)$*. Prove that* ST *is invertible if and only if* S *and* T *are invertible.*

Solution. □

2 For Laura

Problem 4 (Treil 2.3.1)**.** *For what value of b does the system*

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ b \end{pmatrix}$$

have a solution. Find the general solution of the system for this value of b.

Solution. □

Problem 5. *Which of the following are bases for $\mathbb{R}^3$?*

(a) $(1, 2, -1)$, $(1, 0, 2)$, $(2, 1, 1)$

(b) $(-1, 3, 2)$, $(-3, 1, 3)$, $(2, 10, 2)$

Solution. □

Problem 6. *Consider the following vectors in $\mathbb{R}^5$.*

$$v_1 = (2, -1, 1, 5, -3), \quad v_2 = (3, -2, 0, 0, 0), \quad v_3 = (1, 1, 50, -921, 0)$$

(a) Show that these vectors are linearly independent.

(b) Complete v_1, v_2, v_3 to a basis.

Hint: if you do part (b) first, you can do everything without any computation. Hint to the hint: upper triangular matrices.

Solution. □

3 For Beckham

Problem 7. *Determine the dimension of the kernel and the image for the linear maps defined by the following matrices.*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution. □

Problem 8. *A is a 54×37 matrix with rank 31. What are the dimensions of* $\ker A$, $\ker A^t$, $\operatorname{Im} A$, *and* $\operatorname{Im} A^t$*?*

Solution. □

Problem 9 (Axler 3.D.3)**.** *Suppose V is finite dimensional, $U \subset V$ is a subspace, and $S \in L(U, V)$ (i.e. $S : U \to V$ is a linear map). Prove there exists an invertible $T \in L(V)$ so that $Tu = Su$ for every $u \in U$ if and only if S is injective.*

Solution. □

4 For Davis

Problem 10 (c.f. Axler 2.A.1)**.** *Let* $v_1, \ldots, v_n$ *be vectors in* V*. Prove that if* $(v_1, \ldots, v_n)$ *spans* V*, then so does* $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$*. You could do this directly, but I want you to prove this using linear maps. Specifically, consider the linear maps* $T : \mathbb{R}^n \to V$ *and* $T' : \mathbb{R}^n \to V$ *defined by* $T(e_i) = v_i$ *and* $T'(e_i) = \begin{cases} v_i - v_{i+1} & i \leq n-1 \\ v_n & i = n \end{cases}$*. What does the hypothesis tell you about* T*? Find an invertible linear map* $S : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $T' = TS$*.*

Solution. □

Problem 11 (Axler 3.D.4)**.** *Suppose* W *is finite dimensional and* $T, T' \in L(V, W)$*. Prove that* $\ker T = \ker T'$ *if and only if there exists an invertible operator* $S \in L(V)$ *so that* $T' = ST$*. Hint: you may not assume* V *is finite dimensional.*

Solution. □

Problem 12. *Let* $T, T' \in L(V, W)$ *with* W *finite-dimensional. Assume that* $\ker T = \ker T'$*. Show that it's possible to choose bases on* V *and* W *in different ways so that the matrix of* T *with respect to one choice of bases is the matrix of* T' *with respect to the other choice of bases.*[1]

Solution. □

[1] This might be confusing – please ask for clarification if you need it.