Homework 5

Math 25a

Due October 19, 2018

Topics covered (lectures 9-10): matrices, rank-nullity, matrix multiplication, invertibility

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Laura

Problem 1. *For each linear transformation, find its matrix with respect to the standard bases.*

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ *defined by* $T(x, y) = (x + 2y, 2x - 5y, 7y)$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ *defined by*

$$T(x, y, z, w) = (x + y + z + w, y - w, x + 3y + 6w)$$

Solution. □

Problem 2. *This problem is about rotation matrices and trig identities.*

(a) *Let* $A_\theta \in M_2(\mathbb{R})$ *be the matrix of rotation of* $\mathbb{R}^2$ *by* $\theta \in [0, 2\pi)$. *Show by matrix multiplication that* $A_\theta A_{-\theta} = I$.

(b) *Observe geometrically that* $A_\theta A_\eta = A_{\theta+\eta}$. *Then use matrix multiplication to deduce the "angle-sum" formulas for* $\sin(\theta + \eta)$ *and* $\cos(\theta + \eta)$.

Solution. □

Problem 3. *Give examples:*

(a) $A, B \in M_2(F)$ *so that* $A + B$ *is not invertible, although* A *and* B *are invertible.*

(b) $A, B \in M_2(F)$ *so that* A, B, *and* $A + B$ *are all invertible.*

Solution. □

2 For Beckham

Problem 4. *Multiplication of a matrix* $A \in M_2(\mathbb{R})$ *and a vector* $v \in \mathbb{R}^2$ *requires 4 multiplications. Let* D *be a matrix in* $M_{2\times 1000}(\mathbb{R})$*, so the columns of* D *give* 1000 *vectors in* $\mathbb{R}^2$*. Let* $A, B \in M_2(\mathbb{R})$*. How many multiplications are required to compute* ABD*? Consider two possibilities:* $A(BD)$ *and* $(AB)D$*.*

Solution. □

Problem 5 (Axler 3.B.6)**.** *Prove that there is no linear map* $T : \mathbb{R}^5 \to \mathbb{R}^5$ *such that* $\operatorname{Im} T = \ker T$*. (Compare with a similar problem on HW4.)*

Solution. □

Problem 6 (Axler 3.B.16)**.** *Suppose there exists a linear map on* V *whose kernel and image are both finite dimensional. Show that* V *is finite dimensional. (Hint: You may not use the rank-nullity theorem.)*

Solution. □

3 For Davis

Problem 7 (Axler 3.B.22)**.** *Assume U and V are finite dimensional and $U \xrightarrow{S} V \xrightarrow{T} W$ are linear maps. Show that* $\dim \ker TS \leq \dim \ker S + \dim \ker T$.

Solution. □

Problem 8 (Axler 3.D.19)**.** *Let $V = \mathrm{Poly}(\mathbb{R})$. Suppose that $T : V \to V$ is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in V$.*

(a) Prove that T is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in V$.

Solution. □

Problem 9. *The trace of a matrix $A \in M_n(F)$ is the sum of the diagonal entries* $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

(a) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Two matrices $A, B \in M_n(F)$ are called similar if $A = CBC^{-1}$ for some invertible $C \in M_n(F)$. Show that similar matrices have the same trace.

(c) Are the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix}$ similar?

Solution. □

4 For Joey

Problem 10 (Axler 3.B.12). *Let V be finite dimensional and let $T : V \to W$ be a linear map. Show there exists a subspace $U \subset V$ so that $U \cap \ker T = \{0\}$ and $\operatorname{Im} T = \{Tu : u \in U\}$.*

Solution. □

Problem 11 (Axler 3.B.30). *Suppose that $S, T : V \to F$ are linear maps so that $\ker S = \ker T$. Prove that there exists a constant $c \in F$ so that $T = cS$.*

Solution. □

Problem 12 (Axler 3.D.7). *Suppose V and W are finite dimensional. Fix $v \in V$ and let*

$$E = \{T \in L(V, W) : Tv = 0\}$$

(a) Show that E is a subspace of $L(V, W)$.

(b) What is E in the case $v = 0$? Assuming $v \neq 0$, use the rank-nullity theorem to compute the dimension of E.

Solution. □