Homework 4

Math 25a

Due October 12, 2018

Topics covered (lecture 8): linear maps, kernel/image

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Beckham

Problem 1 (Axler 3.A.7)**.** *Suppose that V is 1-dimensional. Show that for every linear map $T: V \to V$ there exists $a \in F$ so that $Tv = av$ for all $v \in V$.*

Solution. □

Problem 2. *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that if z is the midpoint of the line segment $[x, y]$ between $x, y \in \mathbb{R}^n$, then $T(z)$ is the midpoint of $[T(x), T(y)]$. Hint: give a formula for the midpoint of a line segment in terms of the endpoints.*

Problem 3. *Consider the bijection $\mathbb{C} \to \mathbb{R}^2$ defined by $x + iy \mapsto (x, y)$. Under this bijection, we can treat $\mathbb{C}$ either as a 1-dimensional complex vector space or as a 2-dimensional real vector space.*

(a) Treating $\mathbb{C}$ as a complex vector space, show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation of $\mathbb{C}$. What is its matrix? [1]

(b) Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, show that the multiplication by $\alpha = a + ib$ is a linear map. What is its matrix?

(c) Define $T(x + iy) = 2x - y + i(x - 3y)$. Show that T does not define a linear map $\mathbb{C} \to \mathbb{C}$ (viewed as a complex vector space), but it does define a linear map $\mathbb{R}^2 \to \mathbb{R}^2$

Solution. □

[1] We'll discuss matrices in the next lecture, but I think you can figure this out before then.

2 For Davis

Problem 4 (Axler 3.A.8)**.** *Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(av) = af(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but f is not linear.*

Solution. □

Problem 5 (Axler 3.B.9-10)**.** *Let $T : V \to W$ be a linear map.*

(a) Show that if T is injective and $v_1, \ldots, v_n$ are linearly independent in V, then $Tv_1, \ldots, Tv_n$ are linearly independent in W.

(b) Show that if T is surjective and $v_1, \ldots, v_n$ span V, then $Tv_1, \ldots, Tv_n$ span W.

Solution. □

Problem 6. *Consider $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x + y$. Find all the linear maps $S : \mathbb{R} \to \mathbb{R}^2$ so that $TS = I$ (here I refers to the identity map $\mathbb{R} \to \mathbb{R}$).*

Solution. □

3 For Joey

Problem 7 (Axler 3.B.20-21)**.** *Assume W is finite dimensional and $T : V \to W$ is linear.*[2]

(a) *Show that if T is injective, then there exists a linear map $S : W \to V$ so that $ST = I_V$.*

(b) *Show that if T is surjective, then there exists a linear map $S : W \to V$ so that $TS = I_V$.*

Solution. □

Problem 8. *Write a formula for each of the maps $\mathbb{R}^3 \to \mathbb{R}^3$, and verify that they are linear.*

(a) *Project every vector onto the xy-plane.*

(b) *Reflect every vector through the xy-plane.*

(c) *Rotate the xy-plane by $\pi/6$, leaving the z-axis fixed.*

Problem 9. *Work out the kernel of the derivative map $D : \mathrm{Poly}(F) \to \mathrm{Poly}(F)$ when $F = \mathbb{Z}/2\mathbb{Z}$. What happens for $F = \mathbb{Z}/p\mathbb{Z}$?* [3]

Solution. □

[2] Compare this with a similar problem from HW1. The key is to define S so that it is linear. Not every S works!
[3] The derivative is the linear map defined on the standard basis by $D(x^i) = i \cdot x^{i-1}$.

4 For Laura

Problem 10 (Axler 3.B.5)**.** *Give an example of a linear map* $T : \mathbb{R}^4 \to \mathbb{R}^4$ *such that* $\operatorname{Im} T = \ker T$.

Solution. □

Problem 11. *Let* $Poly(F)$ *be the vector space of all polynomials with coefficients in* F*, and let* $V = Fun(F, F)$ *be the vector space of all functions* $f : F \to F$*. Define a map of sets* $T : Poly(F) \to V$ *by* $T(p)(a) = p(a)$ *the function mapping* $a \in F$ *to* $p(a) \in F$*.*

(a) *Show that* T *is a linear map.*

(b) *For* $F = \mathbb{R}$ *show that* T *is injective but not surjective.*

(c) *Give an example of a field* F *where* T *is surjective but not injective, and prove your claim.*

Solution. □

Problem 12. *Let* $F = \mathbb{Z}/p\mathbb{Z}$ *for a prime number* p*. What is the probability that a linear map* $T : F^2 \to F^2$ *is a linear isomorphism when randomly choosing out of all such maps? (Note that a linear isomorphism sends a basis of* F^2 *to another basis of* F^2 *by another problem on this assignment.) As* p *increases is one more or less likely to choose a linear isomorphism at random?*

Solution. □