Homework 3

Math 25a

Due September 28, 2018 at 5pm

Topics covered (Lectures 4-6): subspaces, direct sums, span, linear independence, finite dimensionality

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Davis L.

Problem 1 (Axler 1.C.1)**.** *Let F either be $\mathbb{R}$ or $\mathbb{Z}/2\mathbb{Z}$. For each of the following subsets of F^3, determine whether it is a subspace. Be sure to explain your answer.*

(a) $U_1 = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

(b) $U_2 = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

(c) $U_3 = \{(x_1, x_2, x_3) \in F^3 : x_1x_2x_3 = 0\}$.

Solution. □

Problem 2 (Axler 1.C.12)**.** *Let $U, W \subset V$ be two subspaces. Observe that if $U \subset W$, then $U \cup W = W$ is a subspace. Prove the converse: if $U \cup W$ is a subspace, then either $U \subset W$ or $W \subset U$.*

Solution. □

Problem 3 (Axler 2.A.3)**.** *Find a number t so that $(3, 1, 4)$, $(2, -3, 5)$, $(5, 9, t)$ are linearly dependent in $\mathbb{R}^3$. Find a number t so that the vectors are linearly dependent in $(\mathbb{Z}/2\mathbb{Z})^3$.*

Solution. □

2 For Joey F.

Problem 4 (Axler 2.A.1 and 2.A.6). *Let $v_1, \ldots, v_n$ be vectors in V.*

(a) Prove that if $(v_1, \ldots, v_n)$ spans V, then so does

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n).$$

(b) Prove that if $(v_1, \ldots, v_n)$ is linearly independent in V, then so is

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n).$$

Hint: It might help to first study the case $n = 2, 3$.

Solution. □

Problem 5. *What is the smallest subspace of $M_4(F)$ that contains all upper triangular matrices ($a_{ij} = 0$ if $i > j$) and all symmetric matrices ($A = A^t$). What is the largest subspace contained in both of those subspaces?*

Solution. □

Problem 6 (Axler 1.C.7-8). *(a) Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and taking additive inverses, but U is not a subspace of $\mathbb{R}^2$.*

(b) Give an example of a nonempty subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Solution. □

3 For Laura Z.

Problem 7 (Axler 1.C.24). *Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called* even *if it satisfies $f(-x) = f(x)$ and* odd *if it satisfies $f(-x) = -f(x)$. Let U_e be the subset of even functions and U_o be the subset of odd functions.*

(a) Show U_e and U_o are subspaces of V.

(b) Show $V = U_e \oplus U_o$.

Solution. □

Problem 8 (c.f. Axler 1.C.9). *Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called* periodic *with period p if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. For example, for $a \neq 0$, the function $f(x) = \sin(ax)$ is periodic with period $2\pi/a$. Let $V_p \subset V$ be the subset of functions with period p. Note that $V_0 = V$.*

(a) Fix $p \neq 0$. Is V_p a subspace of V?

(b) Is $\bigcup_{p \in \mathbb{Q} \setminus \{0\}} V_p$ a subspace of V?

(c) Is $\bigcup_{p \in \mathbb{R} \setminus \{0\}} V_p$ a subspace of V?

In each case, give either a proof or a counterexample.

Solution. □

Problem 9 (Axler 2.A.5). *Let V be a vector space over $\mathbb{C}$.*

(a) Explain why V can be given the structure of a vector space over $\mathbb{R}$. Show that if V is finite dimensional over $\mathbb{C}$, then it is finite dimensional over $\mathbb{R}$.

(b) Show that for $V = \mathbb{C}$, the vectors $1+i$ and $1-i$ are linearly independent over $\mathbb{R}$, but linearly dependent over $\mathbb{C}$.

Solution. □

4 For Beckham M.

Problem 10 (Axler 2.A.14)**.** *Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \dots$ of vectors in V such that $(v_1, \dots, v_n)$ is linearly independent for every integer n.*

Solution. □

Problem 11. *Let S be a set and let F be a field. Prove that the vector space $V = \{f : S \to F\}$ of all maps from S to F is finite dimensional if and only if S is finite.*

Solution. □

Problem 12. *Let K be a knot drawn in the plane. As discussed in class, consider the different ways to color the strands with three colors (e.g. puce, gamboge, and xanadu).*

(a) Give the set X of different coloring of K the structure of a vector space (i.e. define an addition and scalar multiplication that satisfy the vector space axioms).

(b) Consider the subset $Y \subset X$ of colorings so that every crossing either has all three colors or only one color. Show that Y is a subspace.

(c) Conclude that the number of elements of Y is a power of three!

Solution. □